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Electron conduction along quantizing magnetic fields in neutron star crusts

I. Theory

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Abstract. Transport properties of degenerate relativistic electrons along quantizing magnetic fields in neutron star crusts are considered. A kinetic equation is derived for the spin polarization density matrix of electrons. Its solution does not depend on the choice of basic electron wave functions unlike previous solutions of the traditional kinetic equation for the distribution function. The density matrix formalism shows that one can always reach high accuracy with the traditional method by a proper choice of the basic functions. Electron Coulomb scattering on ions is considered in liquid matter, and on high-temperature phonons or on charged impurities in solid matter. In the solid regime, the Debye – Waller reduction of phonon scattering can strongly enhance the longitudinal thermal or electric conductivity. An efficient numerical method is proposed for calculating the transport properties of electron gas at any magnetic field of practical interest.

Key words: stars: neutron – dense matter – magnetic fields – radiation mechanisms: thermal

1. Introduction

Accurate transport coefficients in neutron star crusts are important for analysing the thermal evolution of neutron stars and evolution of their magnetic fields. In outer crusts of cooling magnetized neutron stars, the heat is mainly transported along the magnetic fields. There exist several competing heat transport mechanisms across the field, but the longitudinal currents are carried mostly by electrons through their scattering on phonons or charged impurities in the solid phase and on ions in the liquid phase. As a rule, the electrons in the crust are strongly degenerate and may be relativistic; the magnetic field can be easily quantizing. Transport properties of the crusts have been studied in a number of papers (e.g., Yakovlev & Kaminker 1994, and references therein).

In the present work, the most important problem of longitudinal electron transport in quantizing magnetic fields is studied with the use of the quantum density matrix formalism, instead of the traditional kinetic equation for the electron distribution function employed in previous studies. The main advantage of the present approach is that it is independent of the choice of the basis of electron states (basis states are not unique due to the electron spin degeneracy).

We consider three main electron scattering mechanisms. The first one is the Coulomb scattering on ions in the liquid or gaseous phase. The second one is the scattering on high-temperature phonons in the solid phase. In the latter case, we take into account the Debye – Waller factor whose importance has been emphasized and proved by Itoh et al. (1984b, 1993) for the non-magnetic case. We show that the effect of this factor is much stronger in quantizing magnetic fields. The third mechanism is the Coulomb scattering on charged impurities in the solid phase, important much below the melting temperature.

The paper is composed as follows. In Sect. 2 we describe the physical conditions of interest, electron scattering potentials, transport coefficients and their expressions in the non-magnetic case. In Sect. 3 we derive a linearized kinetic equation for the density matrix and compare its numerical solutions with the traditional solutions employed in all previous works. Mathematical properties of the new equation are discussed in Appendix A. The effect of the Debye – Waller factor in quantizing magnetic fields is studied in Sect. 4. The results are summarized in Sect. 5. In Appendix B we present new expressions for some intermediate integrals. These expressions ensure efficient computation of the transport properties for the case when many Landau orbitals are occupied. Previous results (Yakovlev 1984, Hernquist 1984, Schaaf 1988, Van Riper 1988) were restricted to 30 Landau orbitals at most.

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2. Basic equations

2.1. Physical conditions

Magnetic fields B in magnetized neutron star crusts are known to range between 10^{11} and 10^{14} G. For temperatures $T \sim (10^6 - 10^8)$ K and densities $\lesssim 10^8$ g cm^{-3} of interest, the Landau quantization of electron motion in a magnetic field can be very important (then the field is called *quantizing*). Consider not too low densities (typically, $\gtrsim 10^5$ g cm^{-3} at $B \sim 10^{12}$ G, see, e.g., Yakovlev 1984, Van Riper 1988) when the electron gas is almost free and the ions are fully ionized. An electron energy is then given by

$$\epsilon_n(p_z) = c\,\left((m_e c)^2 + 2\hbar\omega_B m_e n + p_z^2\right)^{1/2}, \tag{1}$$

where p_z is the electron momentum along the field, $\omega_B = eB/(m_e c)$ is the electron cyclotron frequency, $(-e)$ is the electron charge, m_e is the electron mass, c is the speed of light, and $n = 0, 1, 2, \ldots$ is a Landau quantum number.

The electron number density can be expressed as

$$n_e = \frac{m_e\omega_B}{(2\pi\hbar)^2}\int_{-\infty}^{\infty} dp_z \sum_{n,s} f_0(\epsilon_n(p_z)), \tag{2}$$

where s is a spin quantum number ($s = -1$ for $n = 0$, and $s = \pm 1$ for $n = 1, 2, \ldots$, see Sect. 3),

$$f_0(\epsilon) = \left[\exp\left(\frac{\epsilon - \mu}{k_B T}\right) + 1\right]^{-1} \tag{3}$$

is the Fermi – Dirac distribution function, and μ is the chemical potential (including $m_e c^2$). A fitting formula for calculation of n_e is given in Appendix C.

The state of ions is mainly determined by the ion coupling parameter $\Gamma = (Ze)^2/(ak_B T)$, where Ze is an ion charge, k_B is the Boltzmann constant, $a = (4\pi n_i/3)^{-1/3}$ is the ion sphere radius, and $n_i = n_e/Z$ is the ion number density. If $\Gamma \ll 1$, the ions constitute an ideal gas. For higher Γ, the gas gradually transforms into a strongly coupled Coulomb liquid. The liquid solidifies at the melting temperature $T = T_m$ which corresponds to $\Gamma = 172$ (Nagara et al. 1987). The zero-point quantum vibrations of the Coulomb crystal become comparable with the thermal vibrations at $k_B T \approx \hbar\omega_p$, where $\omega_p = \sqrt{4\pi Z^2 e^2 n_i/m_i}$ is the ion plasma frequency, and m_i is the ion mass. We will not consider superstrong magnetic fields (discussed, e.g., by Yakovlev 1984) which affect the properties of the ion plasma component.

A more detailed description of the physical conditions is given by Yakovlev (1984) and Van Riper (1988).

2.2. Scattering potentials

Consider three important cases when the electron scattering is almost elastic. The first case is the Coulomb scattering on ions in the liquid or gaseous phase ($T > T_m$). The second case is the scattering on high-temperature phonons ($\hbar\omega_p/k_B \lesssim T < T_m$). The third one is the Coulomb scattering on charged impurities in the lattice, which is important for $k_B T \ll \hbar\omega_p$. The impurities represent ions of charge $Z_{imp} \neq Z$ immersed accidentally in lattice sites. Thus our results will cover a wide range of temperatures.

The electron-ion scattering potential $V(\mathbf{r})$ in the liquid or gaseous phase can approximately be taken as a screened Coulomb potential (Yakovlev 1984). Its Fourier image $U(\mathbf{q})$ is given by

$$|U_{ion}(\mathbf{q})|^2 = \left[4\pi Z e^2/(q^2 + r_s^{-2})\right]^2, \tag{4}$$

where r_s is an effective screening length, $r_s^{-2} = r_i^{-2} + r_e^{-2}$. Here r_i and r_e are the screening lengths due to ions and electrons, respectively. In the most important liquid regime ($1 \lesssim \Gamma < 172$), according to Yakovlev (1984), the ion screening length is

$$r_i = a\sqrt{e/6} \approx 0.67a \tag{5}$$

(Hernquist (1984) used a less accurate approximation for r_i). The electron screening length is determined as

$$r_e = \left[4\pi e^2\, \partial n_e/\partial\mu\right]^{-1/2}. \tag{6}$$

When the temperature is low enough ($k_B T \ll \hbar\omega_B^*$ with $\omega_B^* = \omega_B m_e c^2/\mu$), we have

$$(a_m/r_i)^2 = \frac{6}{e}\,(m_e\hbar\omega_B)^{-1/3}\left[\frac{2}{3\pi Z}\sum_{ns}|p_z|\right]^{2/3}, \tag{7}$$

$$(a_m/r_e)^2 = \frac{2\alpha}{\pi}\sum_{ns}\frac{\epsilon}{|p_z|c}, \tag{8}$$

where $a_m = (\hbar c/eB)^{1/2}$ is the magnetic length, $\alpha = e^2/\hbar c$ is the fine-structure constant, and the energy and momentum variables are assumed to be taken on the Fermi surface: $\epsilon = \epsilon_n(p_z) = \mu$. At arbitrary temperature, the fitting formula of Appendix C can be used to calculate n_e and $\partial n_e/\partial\mu$ for estimation of r_i and r_e according to Eqs. (5) and (6).

For scattering on high-temperature phonons in the solid phase, one has

$$|U_{ph}(\mathbf{q})|^2 = \left(\frac{4\pi Z e^2}{q}\right)^2 \frac{r_T^2}{3}\exp\left[-2W(\mathbf{q})\right], \tag{9}$$

where

$$r_T^2 = \frac{3k_B T u_{-2}}{4\pi n_i Z^2 e^2} = \frac{u_{-2}a^2}{\Gamma} \tag{10}$$

is the mean squared thermal displacement of ions, u_{-2} is a numerical factor determined by the phonon spectrum ($u_{-2} = 13$ for the bcc lattice), and e^{-2W} is the Debye – Waller factor. The latter factor is usually negligible for scattering in terrestrial solids (e.g., Davydov 1976), but it is important in dense neutron star matter (Itoh et al. 1984b, 1993). For the high-temperature solids ($k_B T \gtrsim \hbar\omega_p$) of interest, one has (e.g., Itoh et al. 1984a)

$$2W(\mathbf{q}) \approx (r_T q)^2/3. \tag{11}$$

Note that Eq. (9) represents the familiar high-temperature asymptote of the one-phonon scattering potential (Yakovlev & Urpin 1980) multiplied by the Debye – Waller term (Itoh et al. 1984b) to include multiphonon processes.

Finally, the Coulomb scattering on impurities corresponds to (e.g., Yakovlev & Urpin 1980)

$$|U_{\rm imp}(\mathbf{q})|^2 = \left[4\pi(Z_{\rm imp} - Z)e^2/(q^2 + r_{\rm s}^{-2})\right]^2. \tag{12}$$

In this case the screening length $r_{\rm s}$ is most likely determined by the electrons: $r_{\rm s} = r_{\rm e}$. The Coulomb scattering on impurities is very similar to that on ions, and we do not consider them separately in detail. The results for impurities can be obtained from those for ions by replacing

$$Z \to Z_{\rm imp} - Z, \quad n_{\rm i} \to n_{\rm imp}, \tag{13}$$

where $n_{\rm imp}$ is the impurity number density.

Equations (4) and (9) can be conveniently written as

$$|U_{\rm ion}(\mathbf{q})|^2 = \frac{2\pi}{n_{\rm i} l}\left[\frac{2\hbar c/a_{\rm m}}{q^2 + r_{\rm s}^{-2}}\right]^2, \tag{14}$$

$$|U_{\rm ph}(\mathbf{q})|^2 = \frac{\pi}{n_{\rm i} l}\left(\frac{2\hbar c}{q}\right)^2 {\rm e}^{-2W}, \tag{15}$$

where l is a scale length (Yakovlev 1984):

$$l_{\rm ion} = \frac{m_{\rm e} c^2 \hbar\omega_B}{2\pi n_{\rm i} Z^2 e^4}, \qquad l_{\rm ph} = \frac{3}{4\pi n_{\rm i}}\left(\frac{\hbar c}{Z e^2 r_T}\right)^2. \tag{16}$$

For the scattering on impurities, $U_{\rm imp}$ and $l_{\rm imp}$ are obtained from $U_{\rm ion}$ and $l_{\rm ion}$ by using Eq. (13).

2.3. Transport coefficients

Let j and q be the densities of the electric and thermal currents induced by sufficiently weak electric field $\mathscr{E}$ and gradients of temperature T and electron chemical potential μ directed along the magnetic field (along the z-axis). The currents are determined by three transport coefficients σ, β and λ,

$$\begin{aligned} j &= \sigma\left(\mathscr{E} + \frac{1}{e}\frac{\partial\mu}{\partial z}\right) + \beta\frac{\partial T}{\partial z}, \\ q &= -\beta T\left(\mathscr{E} + \frac{1}{e}\frac{\partial\mu}{\partial z}\right) - \lambda\frac{\partial T}{\partial z}, \end{aligned} \tag{17}$$

where σ is the longitudinal electric conductivity.

For practical use, Eq. (17) can be rewritten as

$$\mathscr{E} + \frac{1}{e}\frac{\partial\mu}{\partial z} = \frac{j}{\sigma} - \frac{\beta}{\sigma}\frac{\partial T}{\partial z}, \qquad q = -\frac{\beta}{\sigma}Tj - æ\frac{\partial T}{\partial z}, \tag{18}$$

where β/σ and $æ = \lambda - T\beta^2/\sigma$ are the longitudinal thermopower and thermal conductivity, respectively.

For nearly elastic electron scattering, the kinetic coefficients σ, β, and λ may be expressed as

$$\begin{pmatrix}\sigma \\ \beta \\ \lambda\end{pmatrix} = \int_{m_{\rm e}c^2}^{\infty} \begin{pmatrix} e^2 \\ e(\epsilon-\mu)/T \\ (\epsilon-\mu)^2/T \end{pmatrix} \frac{n_{\rm e}\tau(\epsilon)c^2}{\epsilon}\left(-\frac{\partial f_0}{\partial\epsilon}\right){\rm d}\epsilon, \tag{19}$$

where $\tau(\epsilon)$ is the effective energy-dependent relaxation time for the electrons.

2.4. Non-magnetic electron relaxation times

In the absence of the magnetic field (or for non-quantizing field), the inverse effective relaxation time (effective collision frequency) of an electron with energy ϵ can be presented as

$$\tau^{-1}(\epsilon) = n_{\rm i} v \sigma_{\rm tr}(\epsilon), \tag{20}$$

where v is the electron velocity and $\sigma_{\rm tr}(\epsilon)$ is the transport cross section:

$$\sigma_{\rm tr}(\epsilon) = \int\frac{{\rm d}\Omega}{4\pi}\int {\rm d}\Omega'\sigma(\mathbf{p}\to\mathbf{p}')(1 - \cos\Theta). \tag{21}$$

Here $\mathbf{p}$ and $\mathbf{p}'$ are electron momenta before and after scattering, respectively, ${\rm d}\Omega$ and ${\rm d}\Omega'$ are solid angle elements, Θ is the scattering angle, and $\sigma(\mathbf{p}\to\mathbf{p}')$ is a differential scattering cross section. In the Born approximation,

$$\sigma(\mathbf{p}\to\mathbf{p}') = \frac{|U(\mathbf{q})|^2\epsilon^2}{4\pi^2\hbar^4 c^4}\left(1 - \frac{v^2}{c^2}\sin^2\frac{\Theta}{2}\right). \tag{22}$$

Consider the Coulomb scattering. Let us substitute the Fourier image (4) into Eq. (22). Integrating in Eq. (21), we arrive at the well known result (e.g., Yakovlev and Urpin 1980):

$$\sigma_{\rm tr}(\epsilon) = 4\pi\left(\frac{Ze^2}{pv}\right)^2\Lambda(\epsilon). \tag{23}$$

Here $\Lambda(\epsilon)$ is the Coulomb logarithm (Yakovlev 1980):

$$\begin{aligned}\Lambda(\epsilon) = {} & \frac{1}{2}\left[\ln(1+w) - (1+w^{-1})^{-1}\right] - \\ & \frac{v^2}{2c^2}\left[1 - 2w^{-1}\ln(1+w) + (1+w)^{-1}\right], \end{aligned} \tag{24}$$

and $w \equiv (2pr_{\rm s}/\hbar)^2$. The terms of order w^{-1} are often neglected in $\Lambda(\epsilon)$ (e.g., Yakovlev and Urpin 1980, Yakovlev 1984). However these terms are very significant in Coulomb liquids of carbon and lighter elements.

Analogously, for the scattering on phonons, from Eqs. (9) and (11) we obtain

$$\sigma_{\rm tr}(\epsilon) = \frac{8\pi}{3}\left(\frac{Ze^2}{\hbar v}\right)^2 r_T^2\left(R_1(\epsilon) - R_2(\epsilon)\frac{v^2}{2c^2}\right), \tag{25}$$

where

$$R_1 = \frac{1}{w}\left(1 - {\rm e}^{-w}\right), \quad R_2 = \frac{2}{w^2}\left(1 - {\rm e}^{-w}(1+w)\right), \tag{26}$$

and $w \equiv (4/3)(pr_T/\hbar)^2$. Generally, we have $R_1 < 1$ and $R_2 < 1$ due to the Debye – Waller factor. If $w \ll 1$, the Debye – Waller factor is insignificant, $R_1 = R_2 = 1$, and Eq. (25) reproduces the result of Urpin and Yakovlev (1980).

Finally, one obtains

$$\tau_0^{\rm ion} = \frac{p^2 v}{4\pi\Lambda(\epsilon)Z^2 e^4 n_{\rm i}}, \tag{27}$$

$$\tau_0^{\rm ph} = \frac{3\hbar^2 v}{8\pi r_T^2 Z^2 e^4 n_{\rm i}}\left(R_1(\epsilon) - R_2(\epsilon)\frac{v^2}{2c^2}\right)^{-1}, \tag{28}$$

for the Coulomb and phonon scattering, respectively.

3. Density matrix

3.1. Kinetic equation

Transport properties of a degenerate relativistic electron gas in quantizing magnetic fields were studied by Yakovlev (1984), Hernquist (1984), Van Riper (1988), and Schaaf (1988) on the basis of the linearized kinetic equations for the electron distribution function. However, as noted by Yakovlev (1984), a self-consistent description should be based on the quantum density matrix formalism.

Let us consider a uniform magnetic field **B** along the z-axis and use the Landau gauge of the vector potential: $\mathbf{A} = (-By, 0, 0)$. Quantum states of a free electron in the magnetic field form a complete orthogonal basis. The basic states can be labelled by the quantum numbers $\kappa = (p_x, p_z, n, s)$, where p_x determines the y-coordinate of the guiding center, $y_B = p_x/(m_e\omega_B)$. An explicit solution of the Dirac equation reads (e.g., Sokolov & Ternov 1968)

$$\psi_\kappa(\mathbf{r}) = \frac{\exp[\mathrm{i}(p_x x + p_z z)/\hbar]}{(a_\mathrm{m} L_x L_z)^{1/2}}\,\chi_{ns}(p_z, y - y_B), \tag{29}$$

where L_x and L_z are the normalization lengths, and χ_{ns} can be chosen as

$$\chi_{n,1}(p_z, y) = \frac{a_\mathrm{m}^{-1/2}}{\sqrt{2E(E+1)}}\begin{pmatrix}(E+1)\,\mathscr{H}_{n-1}(y/a_\mathrm{m})\\ 0 \\ P_{\eta n}(E)\,\mathscr{H}_{n-1}(y/a_\mathrm{m}) \\ -\sqrt{2bn}\,\mathscr{H}_n(y/a_\mathrm{m})\end{pmatrix},$$
$$\chi_{n,-1}(p_z, y) = \frac{a_\mathrm{m}^{-1/2}}{\sqrt{2E(E+1)}}\begin{pmatrix}0\\ (E+1)\,\mathscr{H}_n(y/a_\mathrm{m}) \\ -\sqrt{2bn}\,\mathscr{H}_{n-1}(y/a_\mathrm{m}) \\ -P_{\eta n}(E)\,\mathscr{H}_n(y/a_\mathrm{m})\end{pmatrix}. \tag{30}$$

Here $b = \hbar\omega_B/(m_e c^2)$, $E = \epsilon/(m_e c^2)$ and $P_{\eta n}(E) = \eta\,(E^2 - 1 - 2bn)^{1/2}$ are, respectively, the magnetic field, the energy and the longitudinal momentum in the relativistic units, $\eta = \operatorname{sign} p_z$,

$$\mathscr{H}_n(\xi) = \frac{\exp(-\xi^2/2)}{\pi^{1/4}(2^n n!)^{1/2}}H_n(\xi), \tag{31}$$

and $H_n(\xi)$ is a Hermite polynomial.

In the non-relativistic limit, the two lower components of the bispinors (30) are negligible and this basis corresponds to fixed spin projections ($s\hbar/2$) on the z-axis. However one can use another basis:

$$\chi'_{ns} = \chi_{ns}\cos\phi - s\,\chi_{n,-s}\sin\phi\,. \tag{32}$$

It is sufficient to assume that $0 \le \phi \le \pi/2$; ϕ may depend on n and should vanish for $n = 0$. In particular, the choice

$$\phi = \phi_n \equiv \arcsin\left(\frac{1}{2}\left[1 - (E^2-1)^{-1/2}P_{\eta n}(E)\right]\right)^{1/2} \tag{33}$$

yields the basis of states with fixed helicity used by Hernquist (1984) and Schaaf (1988).

Using the distribution function formalism, one generally obtains different conductivities with different basic functions. Yakovlev (1984) employed both $\phi = 0$ and $\phi = \phi_n$ and interpolated between corresponding results assuming that the former choice is appropriate in the non-relativistic limit ($E \sim 1$) while the latter one is adequate in the ultrarelativistic case ($E \gg 1$).

The accuracy of the distribution function calculations can be verified using the density matrix,

$$\rho_{\kappa'\kappa} = \langle\psi_{\kappa'}|\hat\rho|\psi_\kappa\rangle, \tag{34}$$

where $\hat\rho$ is the statistical operator.

Let us consider elastic collisions and assume that the density matrix is diagonal in energy: $\epsilon' = \epsilon$. Furthermore, we assume the diagonality in momenta. We shall show that the latter property is not violated by collisions. The diagonalities in ϵ and p_z lead to the diagonality in n. The diagonality of the density matrix in p_x, p_z allows us to treat the dependence of $\rho_{\kappa'\kappa}$ on x, z parametrically, without using the Wigner transformation. Thus, we can write

$$\rho_{\kappa'\kappa} = \delta_{n'n}\,\delta_{p'_x p_x}\,\delta_{p'_z p_z}\,\rho_{ns's}(x, z, p_x, p_z), \tag{35}$$

where $\delta_{p'_\alpha p_\alpha} \equiv (2\pi\hbar/L_\alpha)\,\delta(p'_\alpha - p_\alpha)$. Analysing the longitudinal transport properties, we can assume that $\hat\rho$ is independent of x. Let us introduce the spin-polarization density matrix,

$$\rho_{\eta ns's}(z, \epsilon) = \int \rho_{ns's}(z, p_x, p_z)\,\frac{L_x}{2\pi\hbar}\,\mathrm{d}p_x. \tag{36}$$

Then we obtain the kinetic equation with the classical left-hand side

$$\left(\frac{\partial}{\partial t} + v_z\frac{\partial}{\partial z} - e\mathscr{E}\frac{\partial}{\partial p_z}\right)\rho_{\eta n s_1 s_2}(z, \epsilon) = \left[\frac{\mathrm{d}\rho_{\eta n s_1 s_2}}{\mathrm{d}t}\right]_\mathrm{c}, \tag{37}$$

where $\mathscr{E}$ is a longitudinal electric field and v_z is the velocity. The density matrix depends on p_z through ϵ and η. The right-hand side of Eq. (37) is the collision integral to be determined from microscopic considerations. Owing to the linearity of the equations which govern the electron wave-function evolution during a scattering event, the collision integral should be linear in $\rho_{\eta n s_1 s_2}$. Thus it can be presented as

$$\left[\frac{\mathrm{d}\rho_{\eta n s_1 s_2}}{\mathrm{d}t}\right]_\mathrm{c} = \sum_{\eta' n' s'_1 s'_2} D_{\eta' n' s'_1 s'_2;\eta n s_1 s_2}\,\rho_{\eta' n' s'_1 s'_2}. \tag{38}$$

The equilibrium density matrix is $\rho^{(0)}_{s_1 s_2}(\epsilon) = f_0(\epsilon)\,\delta_{s_1 s_2}$, and the collision integral vanishes at the equilibrium.

Deviations from the equilibrium in the linear regime can be treated as small perturbations. In the zero-order approximation, the density matrix is equal to $\rho^{(0)}$; it depends on z parametrically through μ and T. In the first-order approximation, it is customary to write

$$\rho_{\eta n s_1 s_2}(z, \epsilon) = \rho^{(0)}_{s_1 s_2}(\epsilon) + \eta l\,\frac{\partial f_0(\epsilon)}{\partial\epsilon}\left[e\mathscr{E} + \frac{\partial\mu}{\partial z} + \frac{\epsilon - \mu}{T}\frac{\partial T}{\partial z}\right]\varphi_{\eta n s_1 s_2}(\epsilon), \tag{39}$$

where l is the scale length defined by Eq. (16). Keeping the zero-order terms on the left-hand side of Eq. (37) we obtain the set of algebraic equations for the non-equilibrium corrections φ:

$$-\frac{l}{|v_z|}\sum_{\eta' n' s_1' s_2'} \eta' D_{\eta' n' s_1' s_2'; \eta n s_1 s_2}\, \varphi_{\eta' n' s_1' s_2'} = \eta\, \delta_{s_1 s_2}. \tag{40}$$

A solution of these equations allows us to calculate the kinetic coefficients σ, β, and λ in the transport relations (17) and so determine the longitudinal electric and thermal conductivities and thermopower.

Using Eq. (39) we can write σ, β and λ in the form of Eq. (19), with the effective relaxation time defined as

$$\tau(\epsilon) = \frac{\epsilon l m_e \omega_B}{2(\pi\hbar c)^2 n_e}\, \Phi(\epsilon), \tag{41}$$

$$\Phi(\epsilon) = \frac{1}{2}\sum_{\eta=\pm 1}\sum_{n=0}^{n_\epsilon}\sum_{s=\pm 1} \varphi_{\eta n s s}(\epsilon). \tag{42}$$

Here $n_\epsilon = \mathrm{Int}(\nu)$ specifies the highest Landau level populated by electrons with energy ϵ, and

$$\nu = \frac{\epsilon - m_e c^2}{\hbar\omega_B}\,\frac{\epsilon + m_e c^2}{2 m_e c^2} = (E^2 - 1)/(2b). \tag{43}$$

Equations (19) and (41) reproduce Eq. (24) of Yakovlev (1984). However now Eq. (42) contains diagonal elements of the density matrix (to be determined from Eq. (40)) instead of the distribution function in the traditional approach. The factors D in Eq. (40) are obtained in Sect. 3.2. They are found to be much more complicated than analogous factors in the traditional equations.

3.2. Collision integral

First let us derive the collision integral for the full density matrix $\rho_{\kappa'\kappa}$. After that the collision integral (38) is obtained by summing over those quantum numbers in which $\rho_{\kappa'\kappa}$ is diagonal.

In the "quasiclassical approach" for the density matrix (Sobelman et al. 1981), interaction of electrons with perturbers (ions or phonons) is described by a scattering potential $V_0(t)$. A wave function of an interacting electron evolves as $\psi(t) = \hat{S}(t, t_0)\psi(t_0)$, where $\hat{S}$ is the scattering operator. Substituting this into Eq. (34) and averaging over collision parameters we obtain

$$\rho_{\kappa_1\kappa_2}(t) = \sum_{\kappa_1'\kappa_2'} \langle S^*_{\kappa_1'\kappa_1}(t) S_{\kappa_2'\kappa_2}(t)\rangle_{\rm pc}\, \rho_{\kappa_1'\kappa_2'}(-\infty), \tag{44}$$

where $S_{\kappa'\kappa} \equiv \langle\kappa'|\hat{S}(t,-\infty)|\kappa\rangle$ is the scattering matrix, and the brackets $\langle\ldots\rangle_{\rm pc}$ denote space averaging over perturber centers $\mathbf{r}_0$ (i.e., space integration with the weight n_i), while the brackets without subscript denote the quantum-mechanical averaging. The sum over κ' includes that over p_x' and p_z', which should be performed according to the correspondence rule

$$\sum_{p_{x,z}} \longleftrightarrow \int \frac{L_{x,z}}{2\pi\hbar}\, \mathrm{d}p_{x,z}. \tag{45}$$

Following the "adiabatic switch on" method (Landau & Lifshitz 1976), we put

$$V_0(t) = V(\mathbf{r} - \mathbf{r}_0)\, \mathrm{e}^{\varepsilon t}, \tag{46}$$

where $\varepsilon \to 0$ is the adiabatic parameter, and $V(\mathbf{r})$ is the actual scattering potential. Now we obtain

$$\left[\frac{\mathrm{d}\rho_{\kappa_1\kappa_2}}{\mathrm{d}t}\right]_{\rm c} = \sum_{\kappa_1'\kappa_2'} D_{\kappa_1'\kappa_2'\kappa_1\kappa_2}\, \rho_{\kappa_1'\kappa_2'}, \tag{47}$$

where

$$D_{\kappa_1'\kappa_2'\kappa_1\kappa_2} = \lim_{\varepsilon\to 0}\left\langle \frac{\mathrm{d}}{\mathrm{d}t}\left[S^*_{\kappa_1'\kappa_1}(t) S_{\kappa_2'\kappa_2}(t)\right]\Big|_{t=0}\right\rangle_{\rm pc}. \tag{48}$$

The scattering matrix elements $S_{\kappa'\kappa}$ can be calculated in the second-order perturbation theory (e.g., Landau & Lifshitz 1976). Let us substitute them into Eqs. (48) and (47). Then the terms linear in V are canceled after averaging over $\mathbf{r}_0$, while the quadratic terms give

$$\begin{aligned} D_{\kappa_1'\kappa_2',\kappa_1\kappa_2} = &-\frac{\pi}{\hbar^2}\sum_\kappa \Big[\delta_{\kappa_2'\kappa_2}\delta(\omega_{\kappa\kappa_1})\langle V^*_{\kappa\kappa_1} V_{\kappa\kappa_1'}\rangle_{\rm pc} + \\ &\delta_{\kappa_1'\kappa_1}\delta(\omega_{\kappa\kappa_2})\langle V^*_{\kappa\kappa_2'} V_{\kappa\kappa_2}\rangle_{\rm pc}\Big] - \\ &\frac{\mathrm{i}}{\hbar^2}\sum_\kappa \Big[\delta_{\kappa_2'\kappa_2}\mathscr{P}\frac{1}{\omega_{\kappa\kappa_1}}\langle V^*_{\kappa\kappa_1} V_{\kappa\kappa_1'}\rangle_{\rm pc} - \\ &\delta_{\kappa_1'\kappa_1}\mathscr{P}\frac{1}{\omega_{\kappa\kappa_2}}\langle V^*_{\kappa\kappa_2'} V_{\kappa\kappa_2}\rangle_{\rm pc}\Big] + \\ &\frac{\pi}{\hbar^2}\langle V^*_{\kappa_1'\kappa_1} V_{\kappa_2'\kappa_2}\rangle_{\rm pc}\Big(\delta(\omega_{\kappa_1'\kappa_1}) + \delta(\omega_{\kappa_2'\kappa_2})\Big) + \\ &\frac{\mathrm{i}}{\hbar^2}\langle V^*_{\kappa_1'\kappa_1} V_{\kappa_2'\kappa_2}\rangle_{\rm pc}\left(\mathscr{P}\frac{1}{\omega_{\kappa_1'\kappa_1}} - \mathscr{P}\frac{1}{\omega_{\kappa_2'\kappa_2}}\right), \end{aligned} \tag{49}$$

where $\omega_{\kappa'\kappa} \equiv (\epsilon' - \epsilon)/\hbar$, and $V_{\kappa'\kappa}$ is the matrix element of the potential V. In deriving Eq. (49) we have taken into account the well known relationship

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon \mp \mathrm{i}\omega} = \pi\delta(\omega) \pm \mathrm{i}\mathscr{P}\frac{1}{\omega}, \tag{50}$$

where $\mathscr{P}$ denotes the Cauchy main part. An averaged binary product in Eq. (49) is equal to

$$\begin{aligned} \langle V^*_{\kappa_1'\kappa_1} V_{\kappa_2'\kappa_2}\rangle_{\rm pc} = &\frac{2\pi n_i}{(L_x L_z)^2}\,\delta(q_{1x} - q_{2x})\delta(q_{1z} - q_{2z})\int \mathrm{d}q_y \times \\ &|U(\mathbf{q})|^2 M^*_{s_1' s_1}(n', p_z'; n_1, p_{1z}; u) M_{s_2' s_2}(n', p_z'; n_2, p_{2z}; u), \end{aligned} \tag{51}$$

where $\mathbf{q} \equiv (\mathbf{p}' - \mathbf{p})/\hbar$, $U(\mathbf{q})$ is the Fourier image of the potential, and

$$M_{s's}(n', p_z'; n, p_z; u) = \int_{-\infty}^{\infty} \chi^+_{n's'}(p_z', y - q_x a_m^2/2)\chi_{ns}(p_z, y + q_x a_m^2/2)\mathrm{e}^{\mathrm{i}q_y y}\mathrm{d}y \tag{52}$$

is the matrix element which actually depends on q_x and q_y only through the variable $u \equiv (q_x^2 + q_y^2)a_m^2/2$ (e.g., Kaminker &

Yakovlev 1981). In Eq. (51) we have set $p'_{1x} = p'_{2x}$, $p'_{1z} = p'_{2z}$, and $n'_1 = n'_2$, since the density matrix is diagonal in these arguments. Due to the same diagonality, $\omega_{\kappa'_1\kappa_1} = \omega_{\kappa'_2\kappa_2}$, and the last term in Eq. (49) vanishes.

Finally, let us sum $D_{\kappa'_1\kappa'_2\kappa_1\kappa_2}$ over n_2, p_{1x}, p_{2x}, and p_{2z} using Eq. (45). Thus we arrive at the collision integral (38), where

$$D_{\eta' n' s'_1 s'_2; \eta n s_1 s_2} = \frac{|v_z|}{l} \left[a_{\eta n s_1 s_2, \eta' n' s'_1 s'_2} - \frac{1}{2} \delta_{\eta'\eta} \delta_{n'n} \left(A_{\eta n s_1 s'} \delta_{s'_1 s'} \delta_{s'_2 s_2} + A^*_{\eta n s' s_2} \delta_{s'_1 s_1} \delta_{s' s_2} \right) \right]. \quad (53)$$

Explicit expressions for the real coefficients a and complex coefficients A depend on the basis. In the basis (30), they are given in Appendix A.

3.3. Algebraic equations for the density matrix

Transformation properties of the coefficients a and A with respect to the reflection $p_z \to -p_z$ or $p'_z \to -p'_z$ allow us to split the system (40) into two independent subsystems for $\eta = 1$ and $\eta = -1$. Using the basis (30), we have

$$\varphi_{\eta nss} = \varphi_{-\eta,nss} \quad \text{and} \quad \varphi_{\eta ns,-s} = -\varphi_{-\eta,ns,-s}. \quad (54)$$

Thus it is sufficient to solve one of the subsystems:

$$\frac{1}{2} \sum_{s'} \left(A_{ns_1s'} \varphi_{ns's_2} + A^*_{ns's_2} \varphi_{ns_1s'} \right) - \sum_{\eta' n' s' s''} \gamma a_{ns_1s_2,\eta' n' s' s''} \, \varphi_{n's's''} = \delta_{s_1 s_2}, \quad (55)$$

where $\gamma = \eta'$ for $s' = s''$, and $\gamma = 1$ otherwise. In Eq. (55) we have set $\eta = 1$ in a, A, and φ.

Retaining diagonal elements of φ and setting $s_1 = s_2$ in Eq. (55) we recover the algebraic equations derived by Yakovlev (1984) in the distribution function formalism. However thus reduced system is not covariant with respect to the basis transformations (32) (note that the coefficients a and A have been obtained from the matrix elements (52), and they undergo the transformations together with the density matrix elements). The lack of covariance leads to the dependence of the electron transport coefficients on the basis. For instance, Yakovlev (1984) obtained the difference up to 20% using different basis sets for some particular electron gas parameters. Our density matrix formalism makes the complete system (40) covariant with respect to the basis transformations. The trace of the density matrix is invariant, therefore changing the basis does no more affect $\Phi(\epsilon)$ (Eq. (42)) and the kinetic coefficients.

3.4. Applicability range

Let us discuss briefly the validity of the assumptions which led us to the algebraic system (55), for the neutron star crust conditions. First, we have assumed diagonality of the density matrix in p_x and p_z. This property is practically exact. Indeed, the delta-functions $\delta(q_{1x} - q_{2x})\,\delta(q_{1z} - q_{2z})$ in Eq. (51) enter the right-hand side of Eq. (47) through Eq. (49). Since $\rho_{\kappa'_1\kappa'_2}$ is diagonal in p'_x and p'_z, the right-hand side of Eq. (47) virtually contains $\delta(p_{1x} - p_{2x})\,\delta(p_{1z} - p_{2z})$. Thus the diagonality of the density matrix in p_x and p_z is not affected by collisions. Note that the delta-functions in Eq. (51) appeared due to the infinitely large volume assumed in averaging over $\mathbf{r}_0$. If the volume were finite, a non-diagonality in the momenta occurred in the band $\Delta p \sim \hbar/L$, in agreement with the uncertainty principle. An actual value of L is restricted by the condition of spatial uniformity of the considered bulk of matter. The broadening Δp is negligible compared with the typical momentum difference between adjacent Landau orbitals, provided that $L \gg \hbar/(m_e c\sqrt{b})$, which condition is always satisfied in the magnetized neutron star crust.

Another assumption is concerned with the diagonality in the Landau numbers. The last term in Eq. (49) shows that this property is not conserved in collisions. This effect can be estimated with the aid of the uncertainty principle. The non-diagonality should occur within the collisional band $\Delta\epsilon \sim \hbar/\tau$, where τ is the effective relaxation time. The density matrix is diagonal in n if $\Delta\epsilon < \hbar\omega^*_B$. The electron cyclotron energy $\hbar\omega_B$ ranges from 10^2 to 10^4 Ry (where Ry =13.6 eV is the Rydberg energy), for magnetic fields from 10^{11} to 10^{13} G. For estimating $\Delta\epsilon$, we can use the non-magnetic relaxation times (27), (28) (cf. Yakovlev 1984). If $Z \sim 26$, $T \sim 10^7$ K, and the electrons are mildly relativistic, we obtain $\hbar/\tau \sim \Delta\epsilon \sim 30$ Ry $< \hbar\omega_B$. This estimate relates to the collisional broadening. Other broadening mechanisms, not treated here, are due to non-elasticity of scattering and deviations from the Born approximation. Yakovlev (1984) argued that the two latter types of broadening seem to be unimportant under the considered physical conditions.

While considering the spin number $s = \pm 1$, one may notice that the spin-polarization density matrix $\rho_{\eta ns's}$ is reduced compared to the general case of 4×4 matrix (Berestetskij et al. 1982). It is because we restrict the basis to the electron bispinors χ_{ns}, thus neglecting an admixture of positron states. Of course, this restriction is well justified at the non-relativistic temperatures of interest.

Finally, the assumed degeneracy in the spin number is not exact, owing to the quantum-electrodynamical corrections. The corrections split each Landau level into two sublevels (e.g., Landau & Lifshitz 1982). According to the Schwinger formula, the splitting energy is about $(\alpha/2\pi)\hbar\omega_B \sim (0.1 - 10)$ Ry $< \Delta\epsilon$, for $B = 10^{11} - 10^{13}$ G. Therefore the collisional width of the sublevels exceeds their separation, and they can be considered as degenerate.

Thus Eq. (55) is valid for the typical conditions in neutron star crusts.

3.5. Numerical examples and discussion

We have computed the first-order correction $\varphi_{ns_1s_2}(\epsilon)$ to the density matrix and its trace $\Phi(\epsilon)$ for a wide range of magnetic fields from 10^{11} to 10^{14} G. The coefficients in Eq. (55) have been calculated using the formulae from Appendices A and B. The system (55) has been solved using the *LU* factorization code by Fletcher (1988). We have calculated the density matrix and also the electron distribution function in two representations, with $\phi = 0$ and $\phi = \phi_n$ (Eqs. (30), (32), and (33)). The coefficients of the algebraic systems for the distribution functions have been taken from Yakovlev (1984), and the integrals Q_i which enter these coefficients have been calculated using the formulae of Appendix B.

At relatively low field strengths, the density matrix formalism gives practically the same results as the distribution function approach with the basis (30) ($\phi = 0$), in agreement with anticipation of Yakovlev (1984). The inaccuracy of the latter approach is about 0.1% for $B \lesssim 10^{12}$ G. The distribution function results based on the alternative "fixed helicity" representation ($\phi = \phi_n$), employed by Hernquist (1984) and Schaaf (1988), deviate from the exact results by several per cent. When $n_\epsilon = 1$, the deviations reach 3% for the phonon scattering and 8% for the Coulomb scattering. The deviations decrease at higher n_ϵ.

The discrepancies between different representations are larger for stronger fields. Figure 3.5 shows the results for the iron crust ($Z = 26$) with magnetic fields $B = 10^{13}$ and 10^{14} G. The upper panels display the function $\Phi(\epsilon)$ for the electron energies sufficient to occupy up to ten Landau levels. One can see strong quantum oscillations of $\Phi(\epsilon)$. Sharp dips at integer values of ν are caused by singularities of the electron density of states due to the magnetic quantization, when electrons start to populate new Landau levels.

The lower panels show relative errors of the distribution function approximation for two basis choices discussed above. The discrepancies reach up to 20% at $B = 10^{14}$ G for the Coulomb scattering, in agreement with results of Yakovlev (1984). However, the energy dependence of the relative errors is non-monotonous, and a simple interpolation between the approximate results proposed by Yakovlev (1984) is not very accurate. The basis (30) provides higher accuracy in most cases, but the alternative basis (32), (33) becomes more appropriate at relativistic energies for phonon scattering.

Generally, the discrepancies between different representations decrease with increasing energy. We have checked numerically, that relative errors do not exceed 1% for $\nu > 50$ even at $B = 10^{14}$ G. Calculations for such high ν are facilitated by semiclassical formulae for the integrals Q_i derived in Appendix B.

Our calculations show that one should use the density matrix formalism for energies $1 < \nu \lesssim 5$ and relativistic magnetic fields $b \gtrsim 1$, if one needs to keep an error within a few percent. Otherwise the distribution function approach with the basis (30) provides the desired accuracy.

For a better comparison with the results of Yakovlev (1984) and Hernquist (1984), the examples in Fig. 3.5 have been calculated with the scattering potentials used by these authors. In particular, the screening length in the liquid regime has been determined by Eqs. (7) and (8). In the solid regime, we have adopted the conditions when the Debye – Waller factor can be neglected. Under these assumptions, $\Phi(\epsilon)$ does not depend on temperature. The effect of finite temperatures on $\Phi(\epsilon)$ is generally small in the liquid regime. For the solid regime, the effect is discussed in the next section.

4. High-temperature conductivity enhancement in the solid phase

As seen from Eqs. (9) – (11), the Debye – Waller reduction of electron-phonon scattering becomes important at sufficiently high temperatures. Its influence on the electron transport in dense non-magnetized stellar matter has been considered by Itoh et al. (1984a, 1993). The effect is most pronounced near the melting point. In this case it can be easily estimated. Let us put $p = p_0$, where $p_0 = \hbar(3\pi^2 n_e)^{1/3}$ is the field-free Fermi momentum. Taking into account Eq. (10), we can write w in Eq. (26) as $w = (4/3)u_{-2}(9\pi Z/4)^{2/3}/\Gamma$, which gives $w = 560/\Gamma$ for $u_{-2} = 13$, $Z = 26$. Near the melting point $\Gamma \sim 180$, and $w \sim 3$. Then the reduction factors are $R_1 \sim 0.3$ and $R_2 \sim 0.2$. According to Eq. (28), this leads to the largest increase of the relaxation time by a factor of 3.3 for non-relativistic electrons and 2.5 for ultrarelativistic electrons, in qualitative agreement with numerical results of Itoh et al. (1984a, 1993).

However the Debye – Waller factor has been neglected so far while studying the electron transport in quantizing magnetic fields (Yakovlev 1984, Hernquist 1984, Schaaf 1988, Van Riper 1988). In this case, contrary to the non-magnetic one, the longitudinal momentum p_z is quantized into the discrete values $mc\eta P_{\eta n}$ (see Sect. 3.1), and only scattering events with changing p_z do contribute to the transport processes. Therefore there exists a lowest collisional momentum transfer $\Delta p_{\min}(\epsilon)$. According to Eqs. (A11) and (A12) of Appendix A, the scattering rate acquires an additional exponential reduction argument $\zeta\xi_{\min} = (r_T \Delta p_{\min}/\hbar)^2/3$. The smallest momentum difference reaches $\Delta p_{\min} \sim 2\sqrt{2m_e\hbar\omega_B}$ just below the first Landau threshold, and $\Delta p_{\min} \sim m_e\hbar\omega_B/p_0$ just before each next threshold. Therefore the exponent argument is $\zeta\xi_{\min} \sim w$ for $n_\epsilon = 0$, and $\zeta\xi_{\min} \sim w(m_e\hbar\omega_B/p_0^2)^2$ for high n_ϵ. In the latter case, the additional exponent argument becomes small and unimportant. However, it is significant at low n_ϵ.

Figure 4 shows the Debye – Waller enhancement of the electron relaxation time in quantizing magnetic fields. For each T, B, and electron energy ϵ we have put $\mu = \epsilon$ and calculated the electron number density from Eq. (2) to determine the Debye – Waller exponent argument. The fitting formulae presented in Appendix C facilitate this calculation. Then the coefficients of the algebraic system (55) were computed, using the formulae of Appendices A, B, and the system was solved to obtain the function $\Phi(\epsilon)$, which is proportional to the effective relaxation time $\tau(\epsilon)$, Eq. (42). Its ratio to the same function calculated without the Debye – Waller factor is plotted against the Landau variable ν in Fig. 4. This ratio reveals quantum oscillations in phase with oscillations of the function $\Phi(\epsilon)$ itself, thus increasing the amplitude of the latter ones. For comparison, the non-magnetic enhancement factor $(R_1 - (R_2/2)(v/c)^2)^{-1}$ (cf. Eq. (28)) is shown by dashes. As seen from Fig. 4, the quantizing field makes the Debye – Waller enhancement much stronger. When the electrons populate one or two Landau levels, the relaxation time is enhanced by a factor of up to 30 (compared with the factor of about 3 in the non-magnetic case).

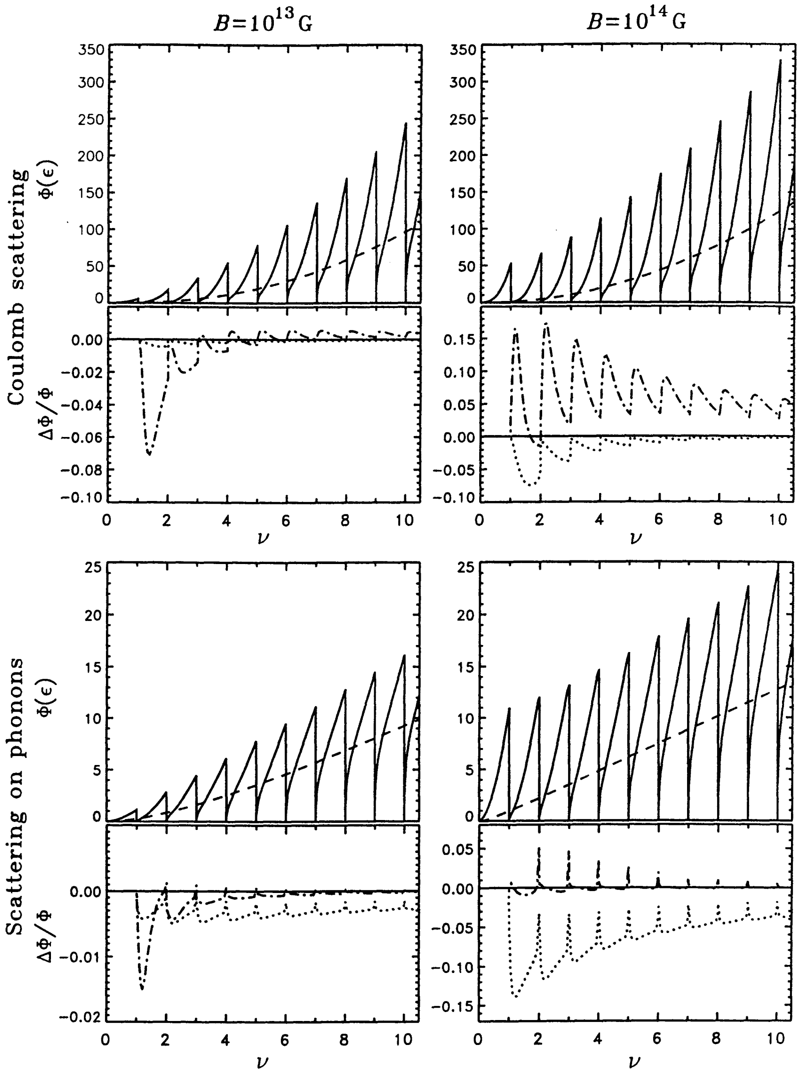

Fig. 1. Function $\Phi(\epsilon)$ given by Eq. (42). Landau variable ν is defined by Eq. (43).
Top panels: $\Phi(\epsilon)$ in the density matrix approach.
Bottom panels: Relative errors $\Delta\Phi/\Phi = (\Phi_{\rm appr}/\Phi - 1)$ of the distribution function approximations $\Phi_{\rm appr}$ (dotted line: basis (30); dot-and-dashed line: basis (32), (33)). Dashes show non-magnetic approximation, according to Eqs. (27), (28), and (41).

5. Summary

We have presented the theory of transport properties of degenerate electrons along quantizing magnetic fields in neutron star crusts. Our results are advanced, compared to the previously known ones, in three respects.

First, a kinetic equation for the spin polarization density matrix of electrons is derived. The solution of this equation provides a justification of the standard approach based on the kinetic equation for the electron distribution function. The present results are compared with two versions of the standard approach used previously by different authors.

For non-relativistic magnetic fields, $B \lesssim 10^{13}$ G, our results confirm the arguments of Yakovlev (1984) that the standard approach which employs basic functions with fixed spin z-projection is the most appropriate in the non-relativistic limit. The fixed-helicity basic functions used by other authors lead to small inaccuracies which however seem to be insignificant in astrophysical implications.

For stronger fields, $B \gtrsim 10^{13}$ G, the inaccuracies of the traditional approach increase up to 20% when density is rather low and the electrons occupy low-lying Landau levels. The density-matrix results allow us to choose the most appropriate version of the standard approach. If density is higher and the electrons populate many Landau levels, the difference between various approaches becomes negligible.

Secondly, we have taken into account the cumulative effect of the Debye – Waller factor with the magnetic quantization. In the non-magnetic case, this factor can increase the thermal and electric conductivities by a factor of 3 just below the melting temperature. We show that the magnetic quantization can enhance the effect by an order of magnitude.

Thirdly, we have derived semiclassical expressions for some intermediate integrals which enter the system of equations either for the density matrix or for a distribution function. These expressions provide fast and accurate calculation of the relaxation time for large number of occupied Landau levels.

In this paper we have calculated the kernel function Φ which should undergo further thermal averaging, Eq. (19), to determine the longitudinal electric and thermal conductivities and

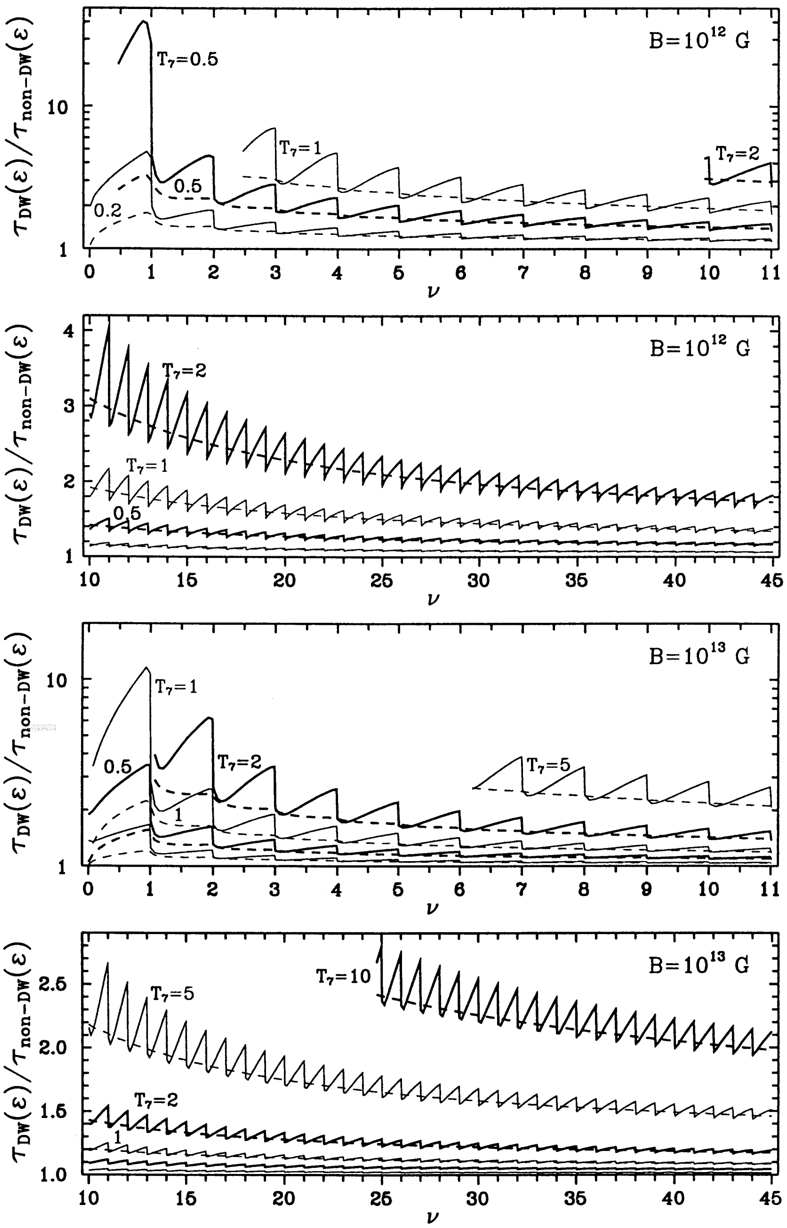

Fig. 2. Ratios of the effective relaxation times $\tau_{\rm DW}(\epsilon)$ to $\tau_{\rm non-DW}(\epsilon)$ calculated, respectively, with and without the Debye – Waller factor.
The temperature $T = T_7 \times 10^7$ K, with T_7 = 0.2, 0.5, 1, 2, 5, and 10. Solid lines – numerical results (Eqs. (41), (42)), dashed lines – non-magnetic case (Eqs. (27), (28)). The curves are plotted starting from the melting points.

thermopower. We shall consider this averaging and astrophysical implications of the developed theory in the subsequent paper.

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Appendix A: coefficients in equations for the density matrix

While deriving Eq. (53), let us first reduce Eq. (47) to Eq. (38). For this purpose, let us sum Eq. (47) over p_{2x}, p_{2z}, n_2 and p_{1z} taking into account Eqs. (35), (36), (49) – (52). This yields

$$\left[\frac{{\rm d}\rho_{\eta n s_1 s_2}}{{\rm d}t}\right]_{\rm c} = -\frac{\pi}{\hbar^2}\sum_{\eta' n' p'_z s' s''} \left(K_{s's_1,s's''}\,\rho_{\eta n s'' s_2}(p_z) + K_{s's'',s's_2}\,\rho_{\eta n s_1 s''}(p_z)\right)\,\delta(\omega) - \frac{\rm i}{\hbar^2}\sum_{\eta' n' p'_z s' s''}\left(K_{s's_1,s's''}\,\rho_{\eta n s'' s_2}(p_z) - K_{s's'',s's_2}\,\rho_{\eta n s_1 s''}(p_z)\right)\,\mathscr{P}\frac{1}{\omega} +$$

$$\frac{2\pi}{\hbar^2} \sum_{\eta' n' p_z' s' s''} K_{s' s_1, s'' s_2} \rho_{\eta' n' s' s''}(p_z')\, \delta(\omega), \tag{A1}$$

where $\omega = (\epsilon' - \epsilon)/\hbar$ and

$$K_{s' s_1, s'' s_2} \equiv K_{s' s_1, s'' s_2}(n', p_z'; n, p_z) = \frac{n_\mathrm{i}}{(2\pi)^2 L_z} \int \mathrm{d}q_x \mathrm{d}q_y \times$$
$$|U(\mathbf{q})|^2 M^*_{s' s_1}(n', p_z'; n, p_z; u) M_{s'' s_2}(n', p_z'; n, p_z; u). \tag{A2}$$

It is convenient to introduce new integration variable $u = (q_x^2 + q_y^2)\, a_\mathrm{m}^2/2$ and the dimensionless function

$$v(u, \xi) = \left(\frac{n_\mathrm{i} l}{2\pi}\right)^{1/2} (\hbar c a_\mathrm{m})^{-1} |U(\mathbf{q})|. \tag{A3}$$

For the Coulomb scattering,

$$v(u, \xi) = (u + \xi)^{-1}, \quad \xi = (q_z^2 + r_\mathrm{s}^{-2})\, a_\mathrm{m}^2/2, \tag{A4}$$

where r_s is the screening length, and $\hbar q_z = p_z' - p_z$. For the scattering on phonons,

$$v(u, \xi) = (u + \xi)^{-1/2} \mathrm{e}^{-W}, \quad \xi = q_z^2 a_\mathrm{m}^2/2. \tag{A5}$$

The sum over p_z' in Eq. (A1) can be converted into the sum over η' and the integral over ϵ'. The delta-function $\delta(\omega)$ eliminates the integration in the first and the last terms of Eq. (A1). Finally we arrive at Eq. (53) with

$$a_{\eta n s_1 s_2, \eta' n' s_1' s_2'} \equiv a_{\eta n s_1 s_2, \eta' n' s_1' s_2'}(E, E), \tag{A6}$$

$$\mathrm{Re}\, A_{\eta n s_1 s_2} = \sum_{\eta' n' s'} a_{\eta n s_1 s_2, \eta' n' s' s'}, \tag{A7}$$

$$\mathrm{Im}\, A_{\eta n s_1 s_2} = \frac{1}{\pi} \sum_{\eta' n' s'} \mathscr{P} \int_{E_{n'}}^{\infty} \frac{\mathrm{d}E'}{E' - E} a_{\eta n s_1 s_2, \eta' n' s' s'}(E, E'), \tag{A8}$$

where $E_{n'} = \sqrt{1 + 2bn'}$,

$$a_{\eta n s_1 s_2, \eta' n' s' s''}(E, E') = \frac{E E'}{|P P'|} \int_0^\infty \mathrm{d}u\, v^2(u, \xi) \times$$
$$M^*_{s' s_1}(n', p_z'; n, p_z; u) M_{s'' s_2}(n', p_z'; n, p_z; u), \tag{A9}$$

and $P \equiv P_{\eta n}(E)$ is defined in Sect. 3.1. According to Eqs. (52) and (30), the matrix elements $M_{s' s}$ contain the integrals

$$\int_{-\infty}^{\infty} \mathscr{H}_{n'}(y/a_\mathrm{m} - q_x a_\mathrm{m}/2) \mathscr{H}_n(y/a_\mathrm{m} + q_x a_\mathrm{m}/2)\, \mathrm{e}^{\mathrm{i} q_y y} \frac{\mathrm{d}y}{a_\mathrm{m}} =$$
$$\left[(n')!/n!\right]^{1/2} \mathrm{e}^{-u/2} u^{(n-n')/2} L_{n'}^{n-n'}(u) = I_{nn'}(u), \tag{A10}$$

where $I_{nn'}(u)$ is a Laguerre function (Sokolov & Ternov 1968). Therefore $M_{s' s}$ can be expressed in terms of the functions

$$Q_2(\xi, n', n, m) = \int_0^\infty \frac{I_{nn'}^2(u)}{(u + \xi)^m} \mathrm{e}^{-\zeta(u+\xi)}\, \mathrm{d}u, \tag{A11}$$

$$Q_3(\xi, n', n, m) = \int_0^\infty \frac{I_{nn'}(u) I_{n-1, n'-1}(u)}{(u + \xi)^m} \mathrm{e}^{-\zeta(u+\xi)}\, \mathrm{d}u, \tag{A12}$$

and $Q_1 = Q_2(\xi, n' - 1, n - 1, m)$. Here $m = 1$ and $\zeta = 2r_T^2/(3a_\mathrm{m}^2)$ for the scattering on phonons; $m = 2$ and $\zeta = 0$ for the Coulomb scattering. The functions Q_1, Q_2, and Q_3 have been introduced by Yakovlev (1984) for the particular case of $\zeta = 0$. The properties of these functions are analysed in Appendix B.

Using the basis (30), we obtain

$$a_{\eta n s_1 s_2, \eta' n' s' s''}(E, E') = \frac{S(s_1, s_2 | s' s'')}{4(E + 1)(E' + 1)\, |P P'|}, \tag{A13}$$

where

$$S(1, 1|1, 1) = g^2 Q_1 + 4b^2 nn' Q_2 + 4bg\sqrt{nn'}\, Q_3, \tag{A14}$$

$$S(-1, -1| -1, -1) = g^2 Q_2 + 4b^2 nn' Q_1 + 4bg\sqrt{nn'} Q_3, \tag{A15}$$

$$S(1, 1| -1, -1) = 2b\,(n' P^2 Q_1 + n P'^2 Q_2 - 2\sqrt{nn'}\, P P' Q_3), \tag{A16}$$

$$S(-1, -1|1, 1) = 2b\,(n' P^2 Q_2 + n P'^2 Q_1 - 2\sqrt{nn'}\, P P' Q_3), \tag{A17}$$

$$S(-1, 1|1, 1) = S(1, -1|1, 1) = -g\sqrt{2bn}\, P' Q_1 + 2bn'\sqrt{2bn} P Q_2 + \sqrt{2bn'}(gP - 2bnP') Q_3, \tag{A18}$$

$$S(1, 1| -1, 1) = S(1, 1|1, -1) = -g\sqrt{2bn'}\, P Q_1 + 2bn\sqrt{2bn'} P' Q_2 + \sqrt{2bn}(gP' - 2bnP) Q_3, \tag{A19}$$

$$S(-1, 1| -1, -1) = S(1, -1| -1, -1) = g\sqrt{2bn}\, P' Q_2 - 2bn'\sqrt{2bn} P Q_1 - \sqrt{2bn'}(gP - 2bnP') Q_3, \tag{A20}$$

$$S(-1, -1| -1, 1) = S(-1, -1|1, -1) = g\sqrt{2bn'}\, P Q_2 - 2bn\sqrt{2bn'} P' Q_1 - \sqrt{2bn}(gP' - 2bnP) Q_3, \tag{A21}$$

$$S(-1, 1| -1, 1) = S(-1, -1|1, -1) = 2bg\sqrt{nn'}\, (Q_1 + Q_2) + (4b^2 nn' + g^2)\, Q_3, \tag{A22}$$

$$S(-1, 1|1, -1) = S(1, -1| -1, 1) = 2b\sqrt{nn'}\, P P' (Q_1 + Q_2) - 2b\,(n' P^2 + n P'^2) Q_3, \tag{A23}$$

and $g = (E + 1)(E' + 1) + P P'$.

The first two pairs of coefficients (Eqs. (A14), (A15), and (A16), (A17)), are related to diagonal density matrix elements for electron transitions without and with spin flip, respectively. At $E = E'$ they reproduce equations of Yakovlev (1984). The remaining equations (A18) – (A23) present the coefficients at off-diagonal elements of the correction φ to the density matrix in the system (55).

Appendix B: integrals $Q_i(\xi, n', n, m)$

For an efficient computation of the coefficients (A6) – (A8) in Eqs. (55), let us consider the properties of the functions Q_i (Eqs. (A11), (A12)). Since $Q_i(\xi, n', n, m) = Q_i(\xi, n, n', m)$, we assume $n' - n \geq 0$ without loss of generality.

Using the polynomial representation for the Laguerre polynomials, the integrals Q_i can be expressed as

$$Q_2(\xi, n', n, m) = \sum_{j=0}^{2n} (-1)^j \sum_k c_{n'nk} c_{n'n, j-k} \times$$
$$(n' - n + j)!\, Q_2(\xi, n' - n + j, 0, m), \tag{B1}$$

$$Q_3(\xi,n',n,m) = \sum_{j=0}^{2n-1}(-1)^j \sum_k c_{n'nk}c_{n'n,j-k}\frac{n-k}{\sqrt{n'n}} \times (n'-n+j)!\,Q_2(\xi,n'-n+j,0,m), \quad \text{(B2)}$$

$$Q_2(\xi,n',0,1) = (1+\zeta)^{-n'-1}\,\mathrm{e}^{\xi}\,E_{n'+1}(\xi+\zeta\xi), \quad \text{(B3)}$$

$$Q_2(\xi,n',0,2) = \mathrm{e}^{\xi}\left(E_{n'}(\xi)-E_{n'+1}(\xi)\right), \quad \text{(B4)}$$

where $E_i(\xi)$ is an integral exponent which is easily calculated (Abramowitz & Stegun 1972),

$$c_{n'nk} = \frac{\sqrt{n'!n!}}{k!(n-k)!(n'-n+k)!} \quad \text{(B5)}$$

are the coefficients of the polynomial expansion of a Laguerre function $I_{n'n}(u)$, and summation index k runs from $\max(0,j-n)$ to $\min(n,j)$. However this method fails for $n \gtrsim 15$ due to the exponentially increasing round-off errors (Hernquist 1984, Schaaf 1988). Direct integration (Hernquist 1984) is not very efficient because of rapid oscillations of the Laguerre functions at large n. The direct integration is especially undesirable for the density matrix computation, since an outer integration over E' is required in Eq. (A8).

To avoid the above difficulties, we propose to use the transformation

$$(u+\xi)^{-m} = \int_0^\infty \frac{x^{m-1}}{(m-1)!}\,\mathrm{e}^{-(u+\xi)\,x}\mathrm{d}x \quad \text{(B6)}$$

in Eqs. (A11), (A12), and change then the integration order. Let us use the equalities (Gradshtein & Ryzhik 1965, Abramowitz and Stegun 1972)

$$\int_0^\infty u^{n'-n}\mathrm{e}^{-u\,(1+x)}L_n^{n'-n}(u)\,L_{n-j}^{n'-n}(u)\,\mathrm{d}u = \frac{(n'+n-j)!\,x^{2n-j}}{(n'-n)!\,(n-j)!\,n!\,(1+x)^{n'+n-j+1}} \times F\left(-n+j,-n;-n'-n+j;1-1/x^2\right) = \frac{x^j}{(1+x)^{n+n'-j+1}}\sum_{k=0}^{n-j}\frac{(n')!\,(n'-j)!}{(n-j-k)!\,(n'-j-k)!(j+k)!}\frac{x^{2k}}{k!}, \quad \text{(B7)}$$

where F is a Gauss hypergeometric function. Then we obtain

$$Q_{2+j}(\xi,n',n,m) = \sum_{k=0}^{n-j}\frac{\left[n!\,(n')!\,(n-j)!\,(n'-j)!\right]^{1/2}}{k!\,(n-j-k)!\,(n'-j-k)!} \times \frac{1}{(m-1)!\,(j+k)!}\int_\zeta^\infty \frac{x^{2k+j+m-1}}{(1+x)^{n'+n-j+1}}\,\mathrm{e}^{-\xi x}\,\mathrm{d}x. \quad \text{(B8)}$$

The main advantage of the representation (B8) is that all terms are positive monotonous functions of ξ.

Using Eq. (B8), we can easily derive various asymptotes. In particular, if $\zeta = 0$ and $\xi \to \infty$, then from the asymptotic properties of the integral in Eq. (B8) we obtain

$$Q_{2+j}(\xi,n',n,m) \simeq \left[\frac{n!\,(n')!}{(n-j)!\,(n'-j)!}\right]^{1/2}\frac{(j+m-1)!}{j!\,(m-1)!} \times (1/\xi)^{m+j}\left[1-(n+n'-j+1)\,(j+m)/\xi+\ldots\right]. \quad \text{(B9)}$$

The functions Q_i are finite for $n' \neq n$, and they diverge at $\xi \to 0$ for $n' = n$:

$$Q_2(\xi,n,n,1) = -\ln\,\xi - \gamma - \sum_{m=1}^{2n}\frac{1}{m} + \frac{(n!)^2}{(2n)!}\sum_{m=1}^{n}\frac{(2n-2m)!\,(2m-1)!}{[(n-m)!\,m!\,]^2} + O(\xi), \quad \text{(B10)}$$

$$Q_3(\xi,n,n,1) = -\ln\,\xi - \gamma - \sum_{m=1}^{2n-1}\frac{1}{m} + \frac{n!\,(n-1)!}{(2n-1)!}\sum_{m=1}^{n-1}\frac{(2n-2m-1)!\,(2m-1)!}{(n-m)!\,(n-m-1)!\,(m!)^2} + O(\xi), \quad \text{(B11)}$$

$$Q_2(\xi,n,n,2) = \xi^{-1} + (2n+1)\left[\ln\,\xi + \sum_{m=2}^{2n+1}\frac{1}{m} + \gamma\right] + \frac{(n!)^2}{(2n)!}\sum_{m=1}^{n}\frac{(2n-2m+1)!\,(2m-2)!}{[(n-m)!\,m!\,]^2} + O(\xi), \quad \text{(B12)}$$

$$Q_3(\xi,n,n,2) = \xi^{-1} + 2n\left[\ln\,\xi + \sum_{m=2}^{2n}\frac{1}{m} + \gamma\right] + \frac{n!\,(n-1)!}{(2n-1)!}\sum_{m=1}^{n-1}\frac{(2n-2m)!\,(2m-2)!}{(n-m)!\,(n-m-1)!\,(m!)^2} + O(\xi). \quad \text{(B13)}$$

However, collecting the terms with $n' = n$ in Eq. (55), we see that only the finite terms of Eqs. (B10)–(B13) do contribute to the coefficients of the algebraic system, while contributions from the divergent terms (ξ^{-1} and $\ln\xi$) are mutually compensated in the real parts of the coefficients (Eqs. (A6), (A7)), as well as in the integrals (A8) for the imaginary parts. Therefore the coefficients of the density matrix equations are essentially finite, except for the singularities at the Landau thresholds due to the factor $|PP'|$ in the denominators of Eqs. (A9), (A13). These remaining singularities stem from divergencies of the electron density of states at the Landau thresholds, and they are responsible for the quantum oscillations of the relaxation time discussed in Sect. 3.5.

Finally, consider the case $n \gg 1$. Let us make use of the semiclassical approximation of a Laguerre function averaged over oscillations (Kaminker & Yakovlev 1981):

$$\overline{I^2_{nn'}(u)} \approx \left[\pi\sqrt{(u-u_1)\,(u-u_2)}\right]^{-1}, \quad \text{(B14)}$$

where $u_{1,2} = (\sqrt{n'} \mp \sqrt{n})^2$. Using again the transformation (B6), we express the integrals Q_i as

$$Q_2(\xi,n',n,1) = \frac{1}{\pi}\int_0^\pi \frac{\exp\left(-2\zeta\sqrt{n'n}\,\cos\vartheta\right)}{n'+n+\xi+2\sqrt{n'n}\,\cos\vartheta}\,\mathrm{d}\vartheta \quad \text{(B15)}$$

$$= \frac{1}{(n'+n+\xi)}\int_{\zeta(n'+n+\xi)}^\infty \mathrm{d}x\,\mathrm{e}^{-x}I_0\left(\frac{2x\sqrt{n'n}}{n'+n+\xi}\right), \quad \text{(B16)}$$

$$4\sqrt{n'n}\,Q_3(\xi,n',n,1) = (n+n')\left[Q_2(\xi,n',n,1)+ Q_2(\xi,n'-1,n-1,1)\right] + \xi\left[Q_2(\xi,n',n-1,1)+\right.$$

$$Q_2(\xi,n'-1,n,1)\big] - \exp\left(-\zeta(n'+n+\xi-1)\right)\times \left[I_0\left(2\zeta\sqrt{n'(n-1)}\right)+I_0\left(2\zeta\sqrt{(n'-1)n}\right)\right], \quad \text{(B17)}$$

$$Q_i(\xi,n',n,2) = -(\partial/\partial\xi)\,Q_i(\xi,n',n,1), \quad \text{(B18)}$$

where I_0 is a modified Bessel function. In deriving Eq. (B17), we have used the relationship (Kaminker & Yakovlev 1981)

$$4\sqrt{nn'}\,I_{nn'}(u)\,I_{n-1,n'-1}(u) = (n+n')\left(I^2_{nn'}(u)+ I^2_{n-1,n'-1}(u)\right) - u\left(I^2_{n-1,n'}(u)+I^2_{n,n'-1}(u)\right). \quad \text{(B19)}$$

The integral (B16) can be easily calculated using polynomial approximations for $I_0(x)$ (Abramowitz & Stegun 1972). At $2\sqrt{n'n}/(n'+n+\xi) < 0.95$, however, direct integration of Eq. (B15) is more efficient, since only a few integration mesh points are needed then. If $\zeta = 0$, the semiclassical approximation allows us to express Q_i in elementary functions:

$$Q_2(\xi,n',n,1) = [(u_1+\xi)(u_2+\xi)]^{-1/2}, \quad \text{(B20)}$$

$$Q_2(\xi,n',n,2) = (\xi+n+n')\,[(u_1+\xi)(u_2+\xi)]^{-3/2}, \quad \text{(B21)}$$

$$4\sqrt{nn'}\,Q_3(\xi,n',n,1) = (n+n')\,(Q_2(\xi,n',n,1)+Q_2(\xi,n'-1,n-1,1)) + \xi\,(Q_2(\xi,n',n-1,1)+Q_2(\xi,n'-1,n,1)) - 2, \quad \text{(B22)}$$

$$4\sqrt{nn'}\,Q_3(\xi,n',n,2) = -((n+n')\xi+(n'-n)^2)\times \left(Q_2^3(\xi,n'-1,n,1)+Q_2^3(\xi,n',n-1,1)\right) + (n+n')\,(Q_2(\xi,n',n,2)+Q_2(\xi,n'-1,n-1,2)). \quad \text{(B23)}$$

We have checked numerically that errors in calculating $\Phi(\epsilon)$ do not exceed 3%, if we substitute the semiclassical formulae (B15) – (B17) and (B20) – (B23) into Eqs. (A14) – (A23) for $n > 10$. The semiclassical approximation makes the computation much faster.

Appendix C: a fitting formula to electron density

Equation (2) can be represented as

$$n_e = \frac{m_e\omega_B k_B T}{2(\pi\hbar)^2 c}\sum_{n,s} F\left(\frac{\mu-\epsilon_n}{k_B T}, \frac{\epsilon_n}{k_B T}\right), \quad \text{(C1)}$$

where $\epsilon_n \equiv \epsilon_n(0)$ is defined by Eq. (1) with $p_z = 0$, and

$$F(x,y) = \int_0^\infty \frac{e^{t-x}}{(e^{t-x}+1)^2}\sqrt{t(t+2y)}\,dt. \quad \text{(C2)}$$

Evidently,

$$\frac{\partial n_e}{\partial\mu} = \frac{m_e\omega_B}{2(\pi\hbar)^2 c}\sum_{n,s} F'\left(\frac{\mu-\epsilon_n}{k_B T}, \frac{\epsilon_n}{k_B T}\right), \quad \text{(C3)}$$

where $F'(x,y) \equiv \partial F(x,y)/\partial x$.

In the non-relativistic limit we have $y \gg 1$ and $x \ll y$ (or $x < 0$). Then $F(x,y)$ reduces to $\sqrt{y/2}\,\mathcal{F}_{-1/2}(x)$, where $\mathcal{F}_{-1/2}(x)$ is a Fermi integral, for which some useful approximations have been presented by Antia (1993). In the opposite case of $y \ll 1$ we have $F(x,y) = \ln(1+e^x)$.

In the general case of arbitrary y, the following approximation is proposed:

$$F(x,y) = \ln(1+e^x)\,\frac{1+y+\xi+c(y)\,a(\xi)\sqrt{\xi+2y}}{1+\xi+c(y)\,b(\xi)}, \quad \text{(C4)}$$

where $\xi = \ln(1+\exp[x-x_0(y)])$,
$x_0(y) = (1+c_1 y^{c_2})^{-1}$,
$c(y) = c_3 y^{c_4}$,
$a(\xi) = \sqrt{\pi}/2 + (a_1+a_2\xi^2)\sqrt{\xi}$,
$b(\xi) = 1 + a_3\sqrt{\xi} + a_4\xi + a_2\xi^3$,
with the numerical parameters $c_1 = 0.623$, $c_2 = 1.6031$, $c_3 = 0.9422$, $c_4 = 1.7262$, $a_1 = 0.103$, $a_2 = 0.043$, $a_3 = 0.0802$, and $a_4 = 0.2944$. At $y \gg 1$, the right-hand side of Eq. (B4) simplifies to $\xi\sqrt{\xi+2y}\,a(\xi)/b(\xi)$, with $\xi = \ln(1+e^x)$, and depends essentially only on the four constants a_i. An error of the approximation (B4) reaches maximum of 0.57% at $x = 2.9$, $y = 5.9$. We have checked additionally, that the x-derivative of this approximation fits the function $F'(x,y)$, which enters Eq. (B3), with a maximum relative error of 2% .

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