Bulk viscosity of superfluid hyperon stars

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We calculate the bulk viscosity due to nonequilibrium weak processes in superfluid nucleon-hyperon matter of neutron stars. For that, the dissipative relativistic hydrodynamics, formulated eariler [M. E. Gusakov, Phys. Rev. D **76**, 083001 (2007).] for superfluid mixtures, is extended to the case when both nucleons and hyperons are superfluid. It is demonstrated that in the most general case (when neutrons, protons, Λ, and Σ^- hyperons are superfluid), nonequilibrium weak processes generate *16* bulk viscosity coefficients, with only *three* of them being independent. In addition, we correct an inaccuracy in a widely used formula for the bulk viscosity of nonsuperfluid nucleon-hyperon matter.

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I. INTRODUCTION

It is well-known (see, e.g., [1–4]), that neutron stars can be unstable with respect to the emission of gravitational waves. The matter in a pulsating neutron star is not (even locally) in chemical equilibrium. The relaxation towards chemical equilibrium is accompanied by the dissipation of pulsation energy. This process is one of the most important dissipative processes, suppressing the growth of gravitational-wave instability. It can be described by the introduction of an effective bulk viscosity in the hydrodynamic equations (see, e.g., [5]).

The bulk viscosity due to nonequilibrium weak processes in neutron stars was calculated by a number of authors (for a review see, e.g., [6]). It strongly depends on the composition of stellar matter. In this paper, we consider the matter of inner layers of neutron stars, composed of electrons, muons, neutrons, protons, Λ, and Σ^- hyperons (nucleon-hyperon matter). A calculation of the bulk viscosity for nucleon-hyperon matter is complicated by the fact that baryons in such matter can be superfluid [7–10].

The bulk viscosity of superfluid matter was calculated in a number of papers (see, e.g., [11–16]). In these papers, only one bulk viscosity coefficient was studied, analogous to that in nonsuperfluid hydrodynamics. The effects of superfluidity were taken into account only in calculating the reaction rates. However, it is known that there are several bulk viscosity coefficients in hydrodynamics of superfluid liquid [5,17–19].

In a recent paper [17], *four* bulk viscosity coefficients were calculated for the matter composed of superfluid neutrons, superfluid protons, and electrons. It was shown that taking into account three additional bulk viscosity coefficients results in a significant decrease of characteristic damping times of sound modes (approximately, by a factor of 3).

In this paper, we extend the results of Ref. [17] to the case of nucleon-hyperon matter. In particular, we demonstrate how the dissipative hydrodynamics [17] should be modified to describe a possible presence of superfluid hyperons. Next, we show that in the most general case, when baryons of any species are superfluid, nonequilibrium processes of mutual transformations of particles generate *sixteen* bulk viscosity coefficients, only three of them being independent. In addition, we correct an inaccuracy in the expression for the bulk viscosity of nonsuperfluid nucleon-hyperon matter made in Ref. [14] and spread widely in the literature (see, e.g., [20–25]). We calculate the bulk viscosity correctly and compare our results with those of Ref. [14].

The paper is organized as follows. In Sec. II, we calculate the bulk viscosity of nonsuperfluid nucleon-hyperon matter. In Sec. III, we calculate and analyze all *sixteen* bulk viscosity coefficients describing dissipation in superfluid nucleon-hyperon mixtures; the relations between these coefficients are also discussed. Section IV presents a summary.

II. BULK VISCOSITY OF NONSUPERFLUID HYPERON MATTER

In this section, we derive an expression for the bulk viscosity due to nonequilibrium processes of particle transformations in a dense nonsuperfluid matter composed of electrons (e), muons (μ), neutrons (n), protons (p), and hyperons (Λ and Σ^- hyperons). Here and below, the variation δA of some physical quantity A is defined as the difference $A - A_0$, where A_0 is the value of A in thermodynamic equilibrium (when matter is unperturbed).

The most effective weak processes in nucleon-hyperon matter are the following nonleptonic reactions [13,14,26,27]

$$n + n \leftrightarrow p + \Sigma^-, \qquad (1)$$

$$n + p \leftrightarrow p + \Lambda, \qquad (2)$$

$$n + n \leftrightarrow n + \Lambda, \qquad (3)$$

$$n + \Lambda \leftrightarrow \Lambda + \Lambda. \qquad (4)$$

The full thermodynamic equilibrium implies the equilibrium with respect to these reactions,

$$2\mu_{n0} - \mu_{p0} - \mu_{\Sigma 0} = 0, \qquad (5)$$

$$\mu_{n0} - \mu_{\Lambda 0} = 0. \qquad (6)$$

Here μ_{i0} are the chemical potentials of particle species $i = n, p, \Lambda, \Sigma$ taken at equilibrium (correspondingly, μ_i are the chemical potentials in the perturbed matter). Notice that the equilibrium conditions for reactions (2)–(4), coincide.

Leptonic reactions (e.g, direct and modified Urca processes with electrons or muons) are much slower in comparison to the reactions (1)–(4). For "typical" perturbation frequencies (e.g., 10^3–10^4 s^{-1} for radial modes or for r modes of rapidly rotating neutron stars) the leptonic reactions cannot influence substantially the chemical composition of perturbed matter. Hence, the main contribution to the bulk viscosity comes from the nonleptonic reactions (1)–(4). In addition to the processes described above, there is a fast nonleptonic reaction due to the strong interaction of baryons

$$n + \Lambda \leftrightarrow p + \Sigma^-. \qquad (7)$$

In accordance with Ref. [14], we assume that the perturbed matter is always in equilibrium with respect to this reaction,

$$\delta\mu_{\rm fast} \equiv \mu_n + \mu_\Lambda - \mu_p - \mu_\Sigma = 0. \qquad (8)$$

Let us obtain the expression for the bulk viscosity of nonsuperfluid nucleon-hyperon matter. For that, we consider a pulsating nucleon-hyperon matter, slightly perturbed from an equilibrium state (so that one can use the linear perturbation theory). If the reactions (1)–(4) are forbidden, then pulsations are reversible and there is no energy dissipation [notice that, the reaction (7) is open]. We denote the pressure in this case by $P_{\rm eq}$. The presence of the reactions (1)–(4) in the pulsating matter leads to a difference between the real pressure P and $P_{\rm eq}$. We define the bulk viscosity ξ by the formula

$$P - P_{\rm eq} \equiv -\xi {\rm div}(\boldsymbol{u}), \qquad (9)$$

where $\boldsymbol{u}$ is the hydrodynamic velocity of pulsations. Notice that this definition differs from the usually accepted one (see, e.g., Ref. [14]). Usually, instead of $P_{\rm eq}$ in formula (9) it is common to substitute the pressure which would be established in the pulsating matter assuming that there is an equilibrium with respect to all the reactions (i.e., the reactions are very fast). Both these approaches are possible.

Generally, the pressure P and the other thermodynamic quantities depend on six parameters, for example, the number densities n_j, where $j = n, p, \Lambda, \Sigma, e, \mu$ (one can neglect the dependence on temperature in strongly degenerate neutron-star matter, see, e.g., Refs. [17,28,29]). However, these parameters are not all independent because in the nucleon-hyperon matter two conditions should be satisfied: the equilibrium condition (8) with respect to the reaction (7) and the condition of quasineutrality,

$$n_p = n_e + n_\mu + n_\Sigma. \qquad (10)$$

Taking into account that the reactions (1)–(4) and (7) conserve the number of leptons and the leptonic processes are neglected, we obtain that the relative number densities of leptons $x_e \equiv n_e/n_b$ and $x_\mu \equiv n_\mu/n_b$ ($n_{\rm b} \equiv n_n + n_p + n_\Lambda + n_\Sigma$ is the baryon number density) remain constant during the pulsations,

$$\delta x_e = \delta x_\mu = 0. \qquad (11)$$

This result is valid only for nonsuperfluid matter and follows from the continuity equations for baryons, electrons, and muons,

$$\frac{\partial \delta n_b}{\partial t} + {\rm div}(n_b \boldsymbol{u}) = 0, \qquad (12)$$

$$\frac{\partial \delta n_e}{\partial t} + {\rm div}(n_e \boldsymbol{u}) = 0, \qquad (13)$$

$$\frac{\partial \delta n_\mu}{\partial t} + {\rm div}(n_\mu \boldsymbol{u}) = 0. \qquad (14)$$

If baryons of any species n, p, Λ, and/or Σ are superfluid, then the continuity equation for baryons (12) should be modified (see Sec. III) and Eq. (11) does not hold.

In view of Eqs. (8) and (10), pressure is a function of only four independent variables, say, n_b, n_H, x_e, and x_μ ($n_H \equiv n_\Lambda + n_\Sigma$ is the hyperon number density). Expanding $P(n_b, n_H, x_e, x_\mu)$ in the Taylor series near the equilibrium state, one obtains for the variation of pressure δP,

$$\delta P = \frac{\partial P(n_b, n_H, x_e, x_\mu)}{\partial n_b} \delta n_b + \frac{\partial P(n_b, n_H, x_e, x_\mu)}{\partial n_H} \delta n_H, \qquad (15)$$

where we used Eq. (11). The variations δn_b and δn_H can be found from the continuity equations for baryons (12) and hyperons,

$$\frac{\partial \delta n_H}{\partial t} + {\rm div}(n_H \boldsymbol{u}) = \Delta\Gamma_1 + \Delta\Gamma_2 + \Delta\Gamma_3 + \Delta\Gamma_4. \qquad (16)$$

Here $\Delta\Gamma_1$, $\Delta\Gamma_2$, $\Delta\Gamma_3$, and $\Delta\Gamma_4$ are the net numbers of hyperons generated in a unit volume per unit time in reactions (1)–(4), respectively. If deviation from the equilibrium state is small, the sources $\Delta\Gamma_l$ ($l = 1, \ldots, 4$) can be expressed as (see, e.g, [11–13])

$$\Delta\Gamma_l = \lambda_l \delta\mu_l, \qquad (17)$$

where λ_l are the "reaction rates," some functions of num-

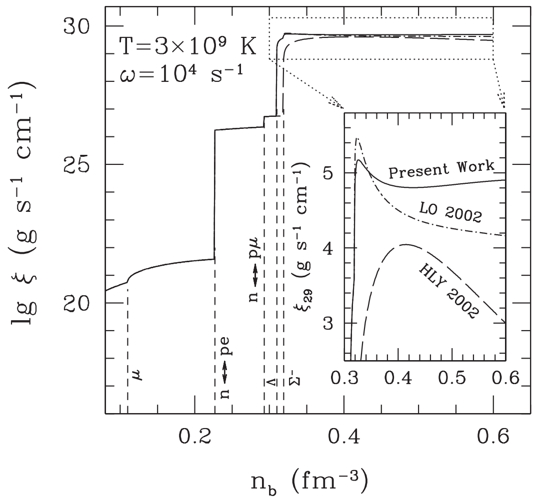

FIG. 1. Bulk viscosity ξ versus baryon number density n_b at $T = 3 \times 10^9$ K and $\omega = 10^4$ s^{-1} for nonsuperfluid matter. Solid, long-dashed, and dot-dashed lines show our results and the results of Ref. [13,14], respectively. Vertical dashes indicate the thresholds for (from left to right): appearance of muons; direct Urca processes involving electrons and muons, respectively; appearance of Λ and Σ^- hyperons, respectively. The inset demonstrates the difference between our calculations and those of Refs. [13,14] in more detail.

ber densities and temperature; $\delta\mu_1 \equiv 2\mu_n - \mu_p - \mu_\Sigma$, $\delta\mu_2 = \delta\mu_3 = \delta\mu_4 \equiv \mu_n - \mu_\Lambda$ are the chemical potential disbalances for the reactions (1)–(4), respectively. Taking into account the equilibrium condition (8) for the fast reaction (7), one has: $\delta\mu_1 = \delta\mu_2 = \delta\mu_3 = \delta\mu_4 \equiv \delta\mu$.

Notice that there is no source in Eq. (16) owing to the fast reaction (7), because this reaction does not change the number of hyperons. The choice of another variable instead of n_H (for example, the neutron number density n_n) would make it necessary to take into account the source due to the reaction (7). In the paper of Lindblom and Owen [14] (and in the subsequent papers [20–25]) the number density of neutrons was chosen as such variable, but the source of neutrons owing to the fast reaction (7) was neglected. This leads to an error in the expression for the bulk viscosity.

In fact, one could think that the source of neutrons due to the fast reaction (7) equals zero, because the matter, as we mentioned before, is in equilibrium with respect to this reaction. However, this is not quite true, because even small (negligible in all other situations) deviation from the equilibrium $\delta\mu_{\rm fast} = \mu_n + \mu_\Lambda - \mu_p - \mu_\Sigma$ multiplied by the large reaction rate $\lambda_{\rm fast}$ results in a finite (nonzero) source $\Delta\Gamma_{\rm fast} = \lambda_{\rm fast}\delta\mu_{\rm fast}$. This fact was emphasized by Jones [26].

Now let us assume that the perturbation of matter is periodic, so that all the thermodynamic quantities oscillate near their equilibrium values with the frequency ω. Then one finds from the continuity equations (12) and (16)

$$\delta n_b = -\frac{n_b}{i\omega}\,{\rm div}(\boldsymbol{u}), \tag{18}$$

$$\delta n_H = -\frac{1}{i\omega}[n_H {\rm div}(\boldsymbol{u}) - \lambda\delta\mu], \tag{19}$$

where $\lambda \equiv \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4$. As in the case of the pressure, the chemical potential disbalance $\delta\mu$ in Eq. (19) is a function of n_b, n_H, x_e, and x_μ. In analogy with Eq. (15) for δP, it can be expanded near the equilibrium state and written as

$$\delta\mu = \frac{\partial\delta\mu(n_b, n_H, x_e, x_\mu)}{\partial n_b}\delta n_b + \frac{\partial\delta\mu(n_b, n_H, x_e, x_\mu)}{\partial n_H}\delta n_H. \tag{20}$$

Here we take into account that $\delta\mu = 0$ in the full equilibrium and that the relative number densities x_e and x_μ do not change in the course of pulsations [see Eq. (11)].

Now, solving the system of equations (18)–(20) and substituting the expressions for δn_b and δn_H into Eq. (15), we derive

$$P - P_{\rm eq} = {\rm div}(\boldsymbol{u})\frac{\partial P(n_b, x_H, x_e, x_\mu)}{\partial x_H}\frac{\lambda}{\omega^2} \times \frac{\partial\delta\mu(n_b, x_H, x_e, x_\mu)}{\partial n_b} \times \left[\frac{i\lambda}{\omega n_b}\frac{\partial\delta\mu(n_b, x_H, x_e, x_\mu)}{\partial x_H} + 1\right]^{-1}. \tag{21}$$

Here the independent variables are n_b, $x_H \equiv n_H/n_b$, x_e, and x_μ. It is easy to express the bulk viscosity ξ from this equation. We are mainly interested in the real part of ξ because it is ${\rm Re}(\xi)$ that is responsible for the energy dissipation (see, e.g., [11]). In this sense it is probably more appropriate to define ${\rm Re}(\xi)$ as the "real" bulk viscosity. It equals

$${\rm Re}\xi = -\frac{n_b^2}{\lambda}\frac{\partial P(n_b, x_H, x_e, x_\mu)}{\partial x_H}\frac{\partial\delta\mu(n_b, x_H, x_e, x_\mu)}{\partial n_b} \times \left[\frac{\partial\delta\mu(n_b, x_H, x_e, x_\mu)}{\partial x_H}\right]^{-2}\frac{1}{1+\omega^2\tau^2}, \tag{22}$$

where $\tau \equiv n_b/\lambda[\partial\delta\mu(n_b, x_H, x_e, x_\mu)/\partial x_H]^{-1}$.

For comparison, we present here the result of Lindblom and Owen [14] [notice that, these authors neglected the reactions (3) and (4), thus assuming that $\lambda_3 = \lambda_4 = 0$]

$$\mathrm{Re}\xi_{\mathrm{L}} = -\frac{n_b^2}{2\lambda_1+\lambda_2}\frac{\partial P(n_b,x_n,x_e,x_\mu)}{\partial x_n} \times \frac{\partial\delta\mu(n_b,x_n,x_e,x_\mu)}{\partial n_b}\left[\frac{\partial\delta\mu(n_b,x_n,x_e,x_\mu)}{\partial x_n}\right]^{-2} \times \frac{1}{1+\omega^2\tau_L^2}, \quad (23)$$

where $\tau_{\mathrm{L}} \equiv n_b/(2\lambda_1+\lambda_2)\ [\partial\delta\mu(n_b,x_n,x_e,x_\mu)/\partial x_n]^{-1}$; $x_n \equiv n_n/n_b$.

The bulk viscosity (22) depends on the reaction rates λ_1, λ_2, λ_3, and λ_4. Some of them were calculated in a number of papers [13,14,20,26,27]. Different authors present different results for the rates; the discussion of advantages and disadvantages of their calculations can also be found in those papers. Unfortunately, to date, there are no strict calculations of the reaction rates. It is reasonable to think that all the rates are of the same order of magnitude.

The dependence of the bulk viscosity on the baryon number density for the temperature $T = 3\times 10^9$ K and the oscillation frequency $\omega = 10^4\ \mathrm{s}^{-1}$ is presented in Fig. 1. While calculating the bulk viscosity, we used the third equation of state of Glendenning [30].

The solid line illustrates *our* results for the bulk viscosity obtained from Eq. (22). We employed the reaction rates from Ref. [14] and, following that paper, we put the rates of the reactions (3) and (4) equal zero, $\lambda_3 = \lambda_4 = 0$. The dot-dashed line is the bulk viscosity calculated as described in Ref. [14] [see also formula (23)]. We remind the reader that in that paper the relative number density of neutrons x_n was chosen as one of the independent variables. However, the source of neutrons due to the fast reaction (7) was erroneously neglected. As one can see, this mistake does not influence the results significantly [typically, by $\sim$(10–30)%]. By the long dashes we show the results of Ref. [13]. In Ref. [13], only one hyperon reaction was taken into account, namely, the reaction (1). As in Ref. [14], the source due to the fast reaction (7) was neglected and, in addition, the equilibrium condition (8) with respect to this reaction was ignored. Moreover, the authors of Ref. [13] used the nonrelativistic approximation when calculating the rate of the reaction (1). This assumption is not well justified for the baryon number densities in the range $n_b \gtrsim (0.3\text{–}0.6)\ \mathrm{fm}^{-3}$ because of the strong dependence of the reaction rates on n_b (see, e.g., Ref. [14]).

III. BULK VISCOSITY OF SUPERFLUID NUCLEON-HYPERON MATTER

In this section, unless it is otherwise stated, the subscripts i and k refer to baryons (i, $k = n$, p, Λ, Σ). The summation is assumed over repeated baryon indices i and k. The subscript l refers to leptons ($l = e$, μ); the subscript j runs over all particle species ($j = n$, p, Λ, Σ, e, μ).

A. The relativistic hydrodynamics of superfluid nucleon-hyperon mixture

In Ref. [17], the dissipative relativistic hydrodynamics of superfluid mixtures was formulated for npe matter. Here we extend this hydrodynamics to the case of superfluid nucleon-hyperon matter composed of superfluid protons, neutrons, Λ and Σ^- hyperons, as well as normal electrons and muons.

The general formulas [Eqs. (26–34) of Ref. [17]] describing the relativistic hydrodynamics of superfluid mixture remain valid with the notion that now the subscripts i and k refer not only to superfluid nucleons (i, $k = n$, p) but also to superfluid hyperons (i, $k = n$, p, Λ, Σ). For instance, the continuity equations for particle species j are written as

$$\partial_\mu j^\mu_{(j)} = 0, \quad (24)$$

with

$$j^\mu_{(i)} = n_i u^\mu + Y_{ik} w^\mu_{(k)}, \qquad j^\mu_{(l)} = n_l u^\mu. \quad (25)$$

Energy-momentum conservation law has the form

$$\partial_\mu T^{\mu\nu} = 0, \quad (26)$$

where

$$T^{\mu\nu} = (P+\varepsilon)u^\mu u^\nu + P\eta^{\mu\nu} + Y_{ik}[w^\mu_{(i)}w^\nu_{(k)} + \mu_i w^\mu_{(k)}u^\nu + \mu_k w^\nu_{(i)}u^\mu] + \tau^{\mu\nu} \quad (27)$$

and $\tau^{\mu\nu}$ is the dissipative correction to the energy-momentum tensor which will be specified below. It satisfies the constraint

$$u_\mu u_\nu \tau^{\mu\nu} = 0. \quad (28)$$

The hydrodynamic equations must be supplemented by the second law of thermodynamics

$$d\varepsilon = TdS + \mu_i dn_i + \mu_e dn_e + \mu_\mu dn_\mu + \frac{Y_{ik}}{2} d[w^\mu_{(i)} w_{(k)\mu}]. \quad (29)$$

Using the quasineutrality condition (10) and the condition of equilibrium (8) with respect to the fast reaction (7), Eq. (29) can be rewritten as

$$d\varepsilon = TdS + \mu_n dn_b - \delta\mu dn_H - (\mu_n - \mu_p - \mu_e)dn_e - (\mu_n - \mu_p - \mu_\mu)dn_\mu + \frac{Y_{ik}}{2} d[w^\mu_{(i)} w_{(k)\mu}], \quad (30)$$

where we remind the reader of the notation $\delta\mu \equiv \mu_n - \mu_\Lambda$. In full thermodynamic equilibrium the third, fourth, and fifth terms are zero because of Eqs. (5) and (6), and of the beta-equilibrium conditions, $\mu_n = \mu_p + \mu_e$ and $\mu_n = \mu_p + \mu_\mu$ (see, e.g., Refs. [11,12]).

In Eqs. (24)–(30) Y_{ik} is a 4×4 symmetric matrix which is related in the nonrelativistic limit to the entrainment matrix ρ_{ik} by the equality [17,31] $Y_{ik} = \rho_{ik}/(m_i m_k)$, where m_i is the mass of a free baryon of species i (the matrix ρ_{ik} is a natural generalization of the superfluid density to the case of superfluid mixtures, see, e.g., Refs. [32–34]). To the best of our knowledge, the matrix Y_{ik} has not been calculated for a nucleon-hyperon matter. Furthermore, ε is the energy density; μ_j is the relativistic chemical potential of particle species j; and S is the entropy density. The pressure P in Eq. (27) is defined in the same way as for ordinary (nonsuperfluid) matter [17,31],

$$P = -\varepsilon + \mu_i n_i + \mu_e n_e + \mu_\mu n_\mu + TS. \tag{31}$$

Next, $\eta^{\mu\nu} = \mathrm{diag}(-1, +1, +1, +1)$ in Eq. (27) is the special relativistic metric; u^μ is the four velocity of normal (nonsuperfluid) liquid component normalized so that $u_\mu u^\mu = -1$ (we assume that all nonsuperfluid components move with the same velocity u^μ). The four-vectors $w^\mu_{(i)}$ satisfy the condition

$$u_\mu w^\mu_{(i)} = 0 \tag{32}$$

and describe motion of superfluid components. To take into account the potentiality of superfluid motion, a four vector $w^\mu_{(i)}$ should be expressed through some scalar functions ϕ_i and written as (see Ref. [17])

$$w^\mu_{(i)} = \partial^\mu \phi_i - q_i A^\mu - (\mu_i + \varkappa_i) u^\mu. \tag{33}$$

Here the scalar ϕ_i is related to the wave function phase of the Cooper-pair condensate Φ_i by the equality $\boldsymbol{\nabla}\phi_i = \hbar \boldsymbol{\nabla}\Phi_i/2$; A^μ is the four potential of the electromagnetic field; q_i is the electric charge of particle species i; $\varkappa_i$ is a small dissipative correction to be determined below.

Note that one can avoid the introduction of new functions ϕ_i in the hydrodynamics of superfluid mixtures if one formulates the potentiality condition (33) in the equivalent way

$$\partial^\nu [w^\mu_{(i)} + q_i A^\mu + (\mu_i + \varkappa_i) u^\mu] = \partial^\mu [w^\nu_{(i)} + q_i A^\nu + (\mu_i + \varkappa_i) u^\nu]. \tag{34}$$

Below we will use the latter formulation because it is more suitable for our purpose. In this approach, four vectors $w^\mu_{(i)}$ are treated as independent hydrodynamic variables.

The hydrodynamics discussed above would be incomplete without an indication of what we mean by a *comoving frame*, that is the frame where we measure (and define) all the thermodynamic quantities. As was demonstrated in Ref. [17], the condition (32) dictates that the comoving is the frame where the four velocity u^μ equals $u^\mu = (1, 0, 0, 0)$. In this frame, the basic thermodynamic quantities ε, n_j, and $\mathbf{w}_{(i)}$ (or $\boldsymbol{\nabla}\phi_i$) are defined by [see Eqs. (25), (27), (28), and (32)]

$$j^0_j = n_j, \tag{35}$$

$$\mathbf{j}_i = Y_{ik} \mathbf{w}_{(k)} = Y_{ik} \boldsymbol{\nabla}\phi_k, \tag{36}$$

$$T^{00} = \varepsilon. \tag{37}$$

All other thermodynamic quantities can be considered as their functions or, equivalently, the functions of ε, n_j, and $w^\mu_{(i)} w_{(k)\mu}$.

In analogy with Ref. [17], from Eqs. (24), (26), and (29), one can derive the entropy generation equation, which defines the dissipative corrections $\tau^{\mu\nu}$ and $\varkappa_i$,

$$\partial_\mu S^\mu = -\frac{\varkappa_i}{T} \partial_\mu [Y_{ik} w^\mu_{(k)}] - \tau^{\mu\nu} \partial_\mu \left(\frac{u_\nu}{T}\right) + Y_{ik} w^\mu_{(k)} \frac{\varkappa_i}{T^2} \partial_\mu T + u^\nu Y_{ik} w^\mu_{(k)} \frac{\varkappa_i}{T} \partial_\nu u_\mu. \tag{38}$$

Here S^μ is the entropy current density,

$$S^\mu = S u^\mu - \frac{u_\nu}{T} \tau^{\mu\nu} - \frac{\varkappa_i}{T} Y_{ik} w^\mu_{(k)}, \tag{39}$$

satisfying the natural constraint $u_\mu S^\mu = -S$. The last two terms in Eq. (38) are small and can be omitted in the majority of applications (for more details, see the discussion in Ref. [17]).

Using the requirement that the entropy does not decrease, one can obtain for the dissipative corrections [neglecting the last two terms in the right-hand side of Eq. (38)]

$$\tau^{\mu\nu} = -\kappa (H^{\mu\gamma} u^\nu + H^{\nu\gamma} u^\mu)(\partial_\gamma T + T u^\delta \partial_\delta u_\gamma) - \eta H^{\mu\gamma} H^{\nu\delta} \left(\partial_\delta u_\gamma + \partial_\gamma u_\delta - \frac{2}{3} \eta_{\gamma\delta} \partial_\varepsilon u^\varepsilon\right) - \xi_{1i} H^{\mu\nu} \partial_\gamma [Y_{ik} w^\gamma_{(k)}] - \xi_2 H^{\mu\nu} \partial_\gamma u^\gamma, \tag{40}$$

$$\varkappa_n = -\xi_{3i} \partial_\mu [Y_{ik} w^\mu_{(k)}] - \xi_{4n} \partial_\mu u^\mu, \tag{41}$$

$$\varkappa_\Lambda = -\xi_{5i} \partial_\mu [Y_{ik} w^\mu_{(k)}] - \xi_{4\Lambda} \partial_\mu u^\mu, \tag{42}$$

$$\varkappa_\Sigma = -\xi_{6i} \partial_\mu [Y_{ik} w^\mu_{(k)}] - \xi_{4\Sigma} \partial_\mu u^\mu, \tag{43}$$

$$\varkappa_p = -\xi_{7i} \partial_\mu [Y_{ik} w^\mu_{(k)}] - \xi_{4p} \partial_\mu u^\mu. \tag{44}$$

Here κ and η are the thermal conductivity and shear viscosity coefficients, respectively; $H^{\mu\nu} \equiv \eta^{\mu\nu} + u^\mu u^\nu$ is the projection matrix; ξ_{1i}, ξ_2, ξ_{3i}, ξ_{4i}, ξ_{5i}, ξ_{6i}, and ξ_{7i} are *25* bulk viscosity coefficients ($i = n, p, \Lambda, \Sigma$). We would like to emphasize that the dissipative corrections (40)–(44) are incomplete in a sense that they (i) do not include various terms related to particle diffusion; (ii) neglect (typically) small terms, explicitly depending on $w^\mu_{(i)}$. For example, we neglect the terms of the form $w^\mu_{(i)} \partial_\mu T$ and $u^\mu w^\nu_{(i)} \partial_\gamma u^\gamma$ in the expressions for $\varkappa_k$ and $\tau^{\mu\nu}$, respectively. The similar approximation is very well-known in the nonrelativistic theory of superfluids (see, e.g., the textbook of Landau and Lifshitz [5], Sec. 140 or Ref. [17]). An inclusion of all these dissipative terms would lead to a number of kinetic coefficients much larger than 27. Since in this paper we are mainly interested in the bulk viscosity coefficients, we restrict ourselves to a simplified form (40)–(44) of dissipative corrections.

The number of bulk viscosity coefficients can be reduced. Notice that, from the quasineutrality condition (10) and the continuity equations (24), it follows that

$$\partial_\mu[Y_{pk} w^\mu_{(k)}] = \partial_\mu[Y_{\Sigma k} w^\mu_{(k)}]. \tag{45}$$

A similar condition for superfluid npe matter was derived in Ref. [17] [see Eq. (41) of that reference]. Owing to the condition (45), there is no need to introduce the bulk viscosity coefficients both for protons and for Σ^- hyperons. For example, it is sufficient to introduce the quantity $\xi_{1\Sigma p} \equiv \xi_{1p} + \xi_{1\Sigma}$ instead of ξ_{1p} and $\xi_{1\Sigma}$. Because of the same reason, we are not interested in the quantities $\varkappa_\Sigma$ and $\varkappa_p$ taken separately. Instead, we will introduce the sum $\varkappa_{\Sigma p} \equiv \varkappa_\Sigma + \varkappa_p$ [notice that, $\varkappa_{\Sigma p}$ is the quantity that appears in the entropy generation equation (38)]. As a result, the corrections $\tau^{\mu\nu}$ and $\varkappa_q$ take the form

$$\tau^{\mu\nu} = -\kappa(H^{\mu\gamma}u^\nu + H^{\nu\gamma}u^\mu)(\partial_\gamma T + T u^\delta \partial_\delta u_\gamma) - \eta H^{\mu\gamma}H^{\nu\delta}\left(\partial_\delta u_\gamma + \partial_\gamma u_\delta - \frac{2}{3}\eta_{\gamma\delta}\partial_\varepsilon u^\varepsilon\right) - \xi_{1q}H^{\mu\nu}\partial_\gamma[Y_{qk}w^\gamma_{(k)}] - \xi_2 H^{\mu\nu}\partial_\gamma u^\gamma, \tag{46}$$

$$\varkappa_n = -\xi_{3q}\partial_\mu[Y_{qk}w^\mu_{(k)}] - \xi_{4n}\partial_\mu u^\mu, \tag{47}$$

$$\varkappa_\Lambda = -\xi_{5q}\partial_\mu[Y_{qk}w^\mu_{(k)}] - \xi_{4\Lambda}\partial_\mu u^\mu, \tag{48}$$

$$\varkappa_{\Sigma p} = -\xi_{6q}\partial_\mu[Y_{qk}w^\mu_{(k)}] - \xi_{4\Sigma p}\partial_\mu u^\mu. \tag{49}$$

Here and below, the subscript q takes on the values n, Λ, and Σp. The expression $\partial_\mu[Y_{\Sigma pk}w^\mu_{(k)}]$ in Eqs. (46)–(49) means $\partial_\mu[Y_{\Sigma k}w^\mu_{(k)}]$ or, equivalently, $\partial_\mu[Y_{pk}w^\mu_{(k)}]$.

As follows from the above equations, we have actually *16* (rather than 25) bulk viscosity coefficients which can contribute to the dissipation of mechanical energy in superfluid nucleon-hyperon matter. Using the Onsager symmetry principle, we obtain

$$\xi_{3\Lambda} = \xi_{5n}, \quad \xi_{3\Sigma p} = \xi_{6n}, \quad \xi_{4n} = \xi_{1n}, \quad \xi_{5\Sigma p} = \xi_{6\Lambda}, \quad \xi_{4\Lambda} = \xi_{1\Lambda}, \quad \xi_{4\Sigma p} = \xi_{1\Sigma p}. \tag{50}$$

Thus, generally, only ten of them are independent.

B. Calculation of the bulk viscosity coefficients for superfluid nucleon-hyperon matter

Let us calculate these phenomenological coefficients assuming they are due to nonequilibrium processes (1)–(4). As in Sec. II, we assume that the matter is slightly perturbed out of equilibrium state and pulsates with the frequency ω. Since the deviation from the equilibrium is small the hydrodynamic equations can be linearized. Because of the same reason, the dependence of various thermodynamic quantities (e.g., the pressure P) on the scalars $w^\mu_{(i)}w_{(k)\mu}$ can be neglected. We assume that the four vectors $w^\mu_{(i)}$ characterizing the superfluid flow of particle species $i = n$, p, Λ, or Σ equal zero in the unperturbed matter. Thus, by perturbing the system, we produce some small $w^\mu_{(i)}$ so that the scalars $w^\mu_{(i)}w_{(k)\mu}$ will be of the second-order smallness and can be omitted in the linear approximation.

We start from the nondissipative relativistic hydrodynamics of superfluid nucleon-hyperon mixture. In this case, the energy-momentum tensor is given by Eq. (27), where the dissipative correction $\tau^{\mu\nu}$ should be set to zero. Similarly, the expressions for $w^\mu_{(i)}$ are given by Eq. (34) with $\varkappa_i = 0$. The nonequilibrium processes (1)–(4) lead to the appearance of the sources in the right-hand side of the continuity equations (24) which, as we will demonstrate below, generate the "effective" dissipative corrections $\tau^{\mu\nu}$ and $\varkappa_q$.

To calculate ξ_{1q} and ξ_2, it is convenient to expand the energy-momentum tensor of the pulsating matter (27) (with $\tau^{\mu\nu} = 0$) in the comoving frame [where $u^\mu = (1, 0, 0, 0)$] near the equilibrium, as it was done in Ref. [17],

$$T^{00} = \varepsilon_0 + \delta\varepsilon, \qquad T^{0m} = T^{m0} = \mu_{i0}Y_{ik}w^m_{(k)}, \qquad T^{nm} = (P_0 + \delta P)\delta_{nm}. \tag{51}$$

Here we restrict ourselves to the linear perturbation terms. The spatial indices n and m vary over 1, 2, and 3; ε_0, μ_{i0}, and P_0 are the corresponding thermodynamic quantities calculated at equilibrium (in the absence of pulsations).

Now our aim is to extract from the tensor (51) various terms which are generated by the nonequilibrium processes (1)–(4) and contribute to dissipation. Then, we will write these terms in the form of a separate dissipative tensor $\tau^{\mu\nu}_{\rm bulk}$.

As follows from Eq. (30), in the linear approximation $\delta\varepsilon$ remains the same as in the absence of dissipation, $\delta\varepsilon = \mu_n\delta n_b$ [this is because the dissipative processes (1)–(4) conserve the number of baryons, hence δn_b is independent of the reaction rates λ_l, $l = 1$, 2, 3, or 4]. Thus, the component $\tau^{00}_{\rm bulk}$ of the tensor $\tau^{\mu\nu}_{\rm bulk}$ is zero. Similarly, $\tau^{m0}_{\rm bulk} = \tau^{0m}_{\rm bulk} = 0$. On the contrary, the variation δP of the pressure contains a dissipative part (we denote it by $\delta P_{\rm dis}$). According to Refs. [11,14] and Sec. II, it is given by

$$\delta P_{\rm dis} = {\rm Re}(\delta P), \tag{52}$$

so that the tensor $\tau^{\mu\nu}_{\rm bulk}$ can be presented in the form (in the comoving frame)

$$\tau^{00}_{\rm bulk} = 0, \qquad \tau^{0m}_{\rm bulk} = \tau^{m0}_{\rm bulk} = 0, \qquad \tau^{nm}_{\rm bulk} = \delta P_{\rm dis}\delta_{nm}. \tag{53}$$

Let us determine $\delta P_{\rm dis}$. For that purpose, we present the pressure P as a function of n_b, n_H, $n_{\Sigma n} \equiv n_\Sigma + n_n$, and

$y \equiv x_e/x_\mu$. All other number densities can be expressed through n_b, n_H, $n_{\Sigma n}$, and y with the help of Eqs. (8) and (10). Notice that we choose $n_{\Sigma n}$ and y instead of the variables x_e and x_μ of Sec. II. The variables x_e and x_μ are less convenient here because Eq. (11) does not hold in the case of superfluid matter. Expanding the pressure $P(n_b, n_H, n_{\Sigma n}, y)$ near the equilibrium state, we write

$$\delta P_{\rm dis} = \frac{\partial P}{\partial n_b}{\rm Re}(\delta n_b) + \frac{\partial P}{\partial n_H}{\rm Re}(\delta n_H) + \frac{\partial P}{\partial n_{\Sigma n}}{\rm Re}(\delta n_{\Sigma n}) + \frac{\partial P}{\partial y}{\rm Re}(\delta y). \tag{54}$$

It is straightforward to show that

$$\delta y = 0, \tag{55}$$

as a consequence of the continuity equations (13) and (14) for electrons and muons. The variations of other variables, n_b, n_H, and $n_{\Sigma n}$, can be found from corresponding continuity equations. Using Eq. (24) and the fact that the variations depend on time t as $\exp(i\omega t)$, we obtain in the comoving frame

$$i\omega\delta n_b + {\rm div}(\boldsymbol{J}_b) = 0, \tag{56}$$

$$i\omega\delta n_H + {\rm div}(\boldsymbol{J}_H) = \Delta\Gamma, \tag{57}$$

$$i\omega\delta n_{\Sigma n} + {\rm div}(\boldsymbol{J}_{\Sigma n}) = -\Delta\Gamma. \tag{58}$$

Here $\Delta\Gamma \equiv \Delta\Gamma_1 + \Delta\Gamma_2 + \Delta\Gamma_3 + \Delta\Gamma_4 = \lambda\delta\mu$, and

$$\delta\mu(n_b, n_H, n_{\Sigma n}) = \frac{\partial\delta\mu}{\partial n_b}\delta n_b + \frac{\partial\delta\mu}{\partial n_H}\delta n_H + \frac{\partial\delta\mu}{\partial n_{\Sigma n}}\delta n_{\Sigma n}. \tag{59}$$

In Eqs. (56)–(58) $\boldsymbol{J}_b \equiv n_b\boldsymbol{u} + \sum_i Y_{ik}\boldsymbol{w}_{(k)}$, $\boldsymbol{J}_H \equiv n_H\boldsymbol{u} + Y_{\Sigma k}\boldsymbol{w}_{(k)} + Y_{\Lambda k}\boldsymbol{w}_{(k)}$, $\boldsymbol{J}_{\Sigma n} \equiv n_{\Sigma n}\boldsymbol{u} + Y_{\Sigma k}\boldsymbol{w}_{(k)} + Y_{nk}\boldsymbol{w}_{(k)}$, $\boldsymbol{u}$ and $\boldsymbol{w}_{(i)}$ are the spatial components of the four velocity u^μ and four vector $w^\mu_{(i)}$, respectively. The solution to the above system of equations gives

$${\rm Re}\,(\delta n_b) = 0, \tag{60}$$

$${\rm Re}\,(\delta n_{\Sigma n}) = -{\rm Re}(\delta n_H), \tag{61}$$

$${\rm Re}\,(\delta n_H) = k\left[\frac{\partial\delta\mu}{\partial n_b}{\rm div}(\boldsymbol{J}_b) + \frac{\partial\delta\mu}{\partial n_H}{\rm div}(\boldsymbol{J}_H) + \frac{\partial\delta\mu}{\partial n_{\Sigma n}}{\rm div}(\boldsymbol{J}_{\Sigma n})\right], \tag{62}$$

where we use the notations $1/k \equiv \lambda(\partial\delta\mu/\partial n_H - \partial\delta\mu/\partial n_{\Sigma n})^2(1 + \omega^2\tau^2)$, $\tau \equiv 1/\lambda(\partial\delta\mu/\partial n_H - \partial\delta\mu/\partial n_{\Sigma n})^{-1}$. Now, using Eq. (54) for $\delta P_{\rm dis}$ and Eqs. (60)–(62), we can find the dissipative tensor $\tau^{\mu\nu}_{\rm bulk}$ in the comoving frame [see Eq. (53)]. Then it can be easily rewritten in an arbitrary frame. The result is

$$\tau^{\mu\nu}_{\rm bulk} = H^{\mu\nu}k\left(\frac{\partial P}{\partial n_H} - \frac{\partial P}{\partial n_{\Sigma n}}\right)\left[\left(n_b\frac{\partial\delta\mu}{\partial n_b} + n_H\frac{\partial\delta\mu}{\partial n_H} + n_{\Sigma n}\frac{\partial\delta\mu}{\partial n_{\Sigma n}}\right)\partial_\gamma u^\gamma + \left(\frac{\partial\delta\mu}{\partial n_b} + \frac{\partial\delta\mu}{\partial n_{\Sigma n}}\right)\partial_\gamma(Y_{nk}w^\gamma_{(k)}) + \left(2\frac{\partial\delta\mu}{\partial n_b} + \frac{\partial\delta\mu}{\partial n_H} + \frac{\partial\delta\mu}{\partial n_{\Sigma n}}\right)\partial_\gamma(Y_{\Sigma pk}w^\gamma_{(k)}) + \left(\frac{\partial\delta\mu}{\partial n_b} + \frac{\partial\delta\mu}{\partial n_H}\right)\partial_\gamma(Y_{\Lambda k}w^\gamma_{(k)})\right]. \tag{63}$$

A comparison of $\tau^{\mu\nu}_{\rm bulk}$ with the phenomenological dissipative tensor $\tau^{\mu\nu}$ [see Eq. (46)] allows us to identify the bulk viscosity coefficients ξ_{1n}, $\xi_{1\Lambda}$, $\xi_{1\Sigma p}$, and ξ_2, generated by nonequilibrium processes (1)–(4)

$$\xi_{1n} = -k\left(\frac{\partial P}{\partial n_H} - \frac{\partial P}{\partial n_{\Sigma n}}\right)\left(\frac{\partial\delta\mu}{\partial n_b} + \frac{\partial\delta\mu}{\partial n_{\Sigma n}}\right), \tag{64}$$

$$\xi_{1\Lambda} = -k\left(\frac{\partial P}{\partial n_H} - \frac{\partial P}{\partial n_{\Sigma n}}\right)\left(\frac{\partial\delta\mu}{\partial n_b} + \frac{\partial\delta\mu}{\partial n_H}\right), \tag{65}$$

$$\xi_{1\Sigma p} = -k\left(\frac{\partial P}{\partial n_H} - \frac{\partial P}{\partial n_{\Sigma n}}\right)\left(2\frac{\partial\delta\mu}{\partial n_b} + \frac{\partial\delta\mu}{\partial n_H} + \frac{\partial\delta\mu}{\partial n_{\Sigma n}}\right), \tag{66}$$

$$\xi_2 = -k\left(\frac{\partial P}{\partial n_H} - \frac{\partial P}{\partial n_{\Sigma n}}\right)\left(n_b\frac{\partial\delta\mu}{\partial n_b} + n_H\frac{\partial\delta\mu}{\partial n_H} + n_{\Sigma n}\frac{\partial\delta\mu}{\partial n_{\Sigma n}}\right). \tag{67}$$

It can be shown that the expression for ξ_2 formally coincides with Eq. (22) for the bulk viscosity of normal matter [however, these formulas give different numerical results because the reaction rates λ_l ($l = 1, \ldots, 4$) differ for superfluid and normal matter].

To prove this, we can rewrite Eq. (67) using n_b, x_H, x_e, and x_μ as independent variables instead of n_b, n_H, $n_{\Sigma n}$, and y. The following equalities will be helpful (f is an arbitrary function)

$$\frac{1}{n_b}\frac{\partial f(n_b, x_H, x_e, x_\mu)}{\partial x_H} = \frac{\partial f(n_b, n_H, n_{\Sigma n}, y)}{\partial n_H} - \frac{\partial f(n_b, n_H, n_{\Sigma n}, y)}{\partial n_{\Sigma n}}, \tag{68}$$

$$n_b\frac{\partial f(n_b, x_H, x_e, x_\mu)}{\partial n_b} = n_b\frac{\partial f(n_b, n_H, n_{\Sigma n}, y)}{\partial n_b} + n_H\frac{\partial f(n_b, n_H, n_{\Sigma n}, y)}{\partial n_H} + n_{\Sigma n}\frac{\partial f(n_b, n_H, n_{\Sigma n}, y)}{\partial n_{\Sigma n}}. \tag{69}$$

To calculate other bulk viscosity coefficients let us apply the same consideration to Eq. (34) for $w^\mu_{(i)}$. As a result, we obtain (in the comoving frame) the expression for the dissipative component $\varkappa_i$ generated by nonequilibrium processes (1)–(4)

$$\varkappa_i = \delta\mu_{i\mathrm{dis}} = \frac{\partial \mu_i}{\partial n_b}\,\mathrm{Re}(\delta n_b) + \frac{\partial \mu_i}{\partial n_H}\,\mathrm{Re}(\delta n_H) + \frac{\partial \mu_i}{\partial n_{\Sigma n}}\,\mathrm{Re}(\delta n_{\Sigma n}). \tag{70}$$

Here $\delta\mu_{i\mathrm{dis}}$ is the dissipative term in the Taylor expansion of the chemical potential μ_i near the equilibrium state (similar to δP_{dis}); $\mathrm{Re}(\delta n_b)$, $\mathrm{Re}(\delta n_H)$, and $\mathrm{Re}(\delta n_{\Sigma n})$ are taken from Eqs. (60)–(62). In a fully covariant form, $\varkappa_q$ is given by Eqs. (47)–(49) with the bulk viscosity coefficients

$$\xi_{3n} = -k\left(\frac{\partial \mu_n}{\partial n_H} - \frac{\partial \mu_n}{\partial n_{\Sigma n}}\right)\left(\frac{\partial \delta\mu}{\partial n_b} + \frac{\partial \delta\mu}{\partial n_{\Sigma n}}\right), \tag{71}$$

$$\xi_{3\Lambda} = -k\left(\frac{\partial \mu_n}{\partial n_H} - \frac{\partial \mu_n}{\partial n_{\Sigma n}}\right)\left(\frac{\partial \delta\mu}{\partial n_b} + \frac{\partial \delta\mu}{\partial n_H}\right), \tag{72}$$

$$\xi_{3\Sigma p} = -k\left(\frac{\partial \mu_n}{\partial n_H} - \frac{\partial \mu_n}{\partial n_{\Sigma n}}\right)\left(2\frac{\partial \delta\mu}{\partial n_b} + \frac{\partial \delta\mu}{\partial n_H} + \frac{\partial \delta\mu}{\partial n_{\Sigma n}}\right), \tag{73}$$

$$\xi_{4n} = -k\left(\frac{\partial \mu_n}{\partial n_H} - \frac{\partial \mu_n}{\partial n_{\Sigma n}}\right)\left(n_b\frac{\partial \delta\mu}{\partial n_b} + n_H\frac{\partial \delta\mu}{\partial n_H} + n_{\Sigma n}\frac{\partial \delta\mu}{\partial n_{\Sigma n}}\right), \tag{74}$$

$$\xi_{4\Lambda} = -k\left(\frac{\partial \mu_\Lambda}{\partial n_H} - \frac{\partial \mu_\Lambda}{\partial n_{\Sigma n}}\right)\left(n_b\frac{\partial \delta\mu}{\partial n_b} + n_H\frac{\partial \delta\mu}{\partial n_H} + n_{\Sigma n}\frac{\partial \delta\mu}{\partial n_{\Sigma n}}\right), \tag{75}$$

$$\xi_{4\Sigma p} = -k\left[\frac{\partial(\mu_p + \mu_\Sigma)}{\partial n_H} - \frac{\partial(\mu_p + \mu_\Sigma)}{\partial n_{\Sigma n}}\right] \times \left(n_b\frac{\partial \delta\mu}{\partial n_b} + n_H\frac{\partial \delta\mu}{\partial n_H} + n_{\Sigma n}\frac{\partial \delta\mu}{\partial n_{\Sigma n}}\right), \tag{76}$$

$$\xi_{5n} = -k\left(\frac{\partial \mu_\Lambda}{\partial n_H} - \frac{\partial \mu_\Lambda}{\partial n_{\Sigma n}}\right)\left(\frac{\partial \delta\mu}{\partial n_b} + \frac{\partial \delta\mu}{\partial n_{\Sigma n}}\right), \tag{77}$$

$$\xi_{5\Lambda} = -k\left(\frac{\partial \mu_\Lambda}{\partial n_H} - \frac{\partial \mu_\Lambda}{\partial n_{\Sigma n}}\right)\left(\frac{\partial \delta\mu}{\partial n_b} + \frac{\partial \delta\mu}{\partial n_H}\right), \tag{78}$$

$$\xi_{5\Sigma p} = -k\left(\frac{\partial \mu_\Lambda}{\partial n_H} - \frac{\partial \mu_\Lambda}{\partial n_{\Sigma n}}\right)\left(2\frac{\partial \delta\mu}{\partial n_b} + \frac{\partial \delta\mu}{\partial n_H} + \frac{\partial \delta\mu}{\partial n_{\Sigma n}}\right), \tag{79}$$

$$\xi_{6n} = -k\left[\frac{\partial(\mu_p + \mu_\Sigma)}{\partial n_H} - \frac{\partial(\mu_p + \mu_\Sigma)}{\partial n_{\Sigma n}}\right]\left(\frac{\partial \delta\mu}{\partial n_b} + \frac{\partial \delta\mu}{\partial n_{\Sigma n}}\right), \tag{80}$$

$$\xi_{6\Lambda} = -k\left[\frac{\partial(\mu_p + \mu_\Sigma)}{\partial n_H} - \frac{\partial(\mu_p + \mu_\Sigma)}{\partial n_{\Sigma n}}\right]\left(\frac{\partial \delta\mu}{\partial n_b} + \frac{\partial \delta\mu}{\partial n_H}\right), \tag{81}$$

$$\xi_{6\Sigma p} = -k\left[\frac{\partial(\mu_p + \mu_\Sigma)}{\partial n_H} - \frac{\partial(\mu_p + \mu_\Sigma)}{\partial n_{\Sigma n}}\right]\left(2\frac{\partial \delta\mu}{\partial n_b} + \frac{\partial \delta\mu}{\partial n_H} + \frac{\partial \delta\mu}{\partial n_{\Sigma n}}\right). \tag{82}$$

In Eqs. (71)–(82), we assumed that the thermodynamic quantities are functions of n_b, n_H, $n_{\Sigma n}$, and y. One can easily check that all of the six Onsager relations (50) are satisfied. Moreover, it turns out that the bulk viscosities (64)–(67) and (71)–(82) obey a number of additional constraints ($q = n, \Lambda, \Sigma p$)

$$\xi_{6q} = \xi_{3q} + \xi_{5q}, \qquad \xi_{4\Sigma p} = \xi_{4n} + \xi_{4\Lambda}, \tag{83}$$

$$\xi_{1n}^2 = \xi_2\xi_{3n}, \qquad \xi_{1\Lambda}^2 = \xi_2\xi_{5\Lambda}, \qquad \xi_{1\Sigma p}^2 = \xi_2\xi_{6\Sigma p}, \tag{84}$$

$$\xi_{1n}\xi_{1\Lambda} = \xi_2\xi_{5n}, \qquad \xi_{1\Sigma p}\xi_{1\Lambda} = \xi_2\xi_{6\Lambda}, \qquad \xi_{1n}\xi_{1\Sigma p} = \xi_2\xi_{6n} \tag{85}$$

so that we have only *three* independent bulk viscosity coefficients, say ξ_2, ξ_{1n}, and $\xi_{1\Lambda}$. All other coefficients can be expressed through these three. The coefficients ξ_2, ξ_{1n}, and $\xi_{1\Lambda}$ are compared in Fig. 2. Because the dimensions of the coefficients ξ_{1i} ($i = n, \Lambda$) and ξ_2 are different, we show the dimensionless combinations $\xi_{1i}n_i/\xi_2$ as functions of the baryon number density n_b.

What is the nature of the relations (83)–(85)? The relations (83) follow from the equilibrium condition (8) with respect to the fast reaction (7). This condition is valid even if we allow for the dissipative processes (1)–(4). Consequently, we can write

$$\mathrm{Re}\,(\mu_\Sigma) + \mathrm{Re}(\mu_p) = \mathrm{Re}(\mu_n) + \mathrm{Re}(\mu_\Lambda) \tag{86}$$

or, in view of Eq. (70) [we remind that $\mathrm{Re}(\mu_i) \equiv \mu_{i\mathrm{dis}}$, $\varkappa_{\Sigma p} \equiv \varkappa_\Sigma + \varkappa_p$],

$$\varkappa_{\Sigma p} = \varkappa_n + \varkappa_\Lambda. \tag{87}$$

Then, substituting Eqs. (47)–(49) into Eq. (87) and equat-

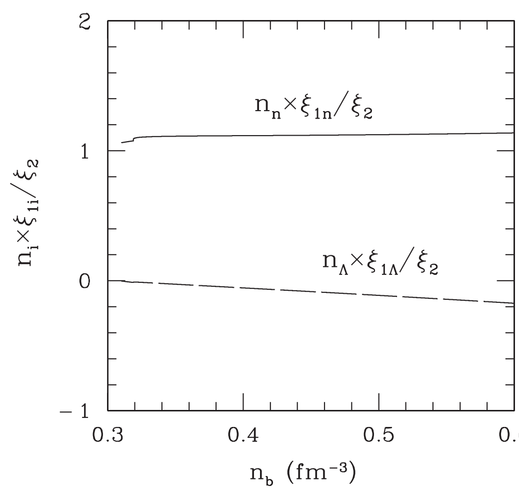

FIG. 2. Dimensionless parameters $\xi_{1i}n_i/\xi_2$ ($i = n, \Lambda$) versus n_b in superfluid nucleon-hyperon matter.

ing prefactors in front of the same four divergences, we obtain the relations (83).

It is convenient to note that the constraint (87) leads to a simple equation interrelating four vectors $w^{\mu}_{(i)}$. It holds true only if the bulk viscosities are generated by the nonequilibrium reactions. To derive this equation, we sum up the potentiality conditions (34) (with proper signs) for all baryon species. Then, using Eqs. (8) and (87), we get

$$\partial^{\nu}[w^{\mu}_{(n)} + w^{\mu}_{(\Lambda)} - w^{\mu}_{(\Sigma)} - w^{\mu}_{(p)}] = \partial^{\mu}[w^{\nu}_{(n)} + w^{\nu}_{(\Lambda)} - w^{\nu}_{(\Sigma)} - w^{\nu}_{(p)}]. \quad (88)$$

In view of this equation and Eq. (45), there are only $4 - 2 = 2$ independent four-vectors $w^{\mu}_{(i)}$ in the system.

Now let us explain why the bulk viscosities satisfy Eqs. (84) and (85). For that purpose, we consider the entropy generation equation. Neglecting all the dissipative processes except for the nonequilibrium reactions (1)–(4) (e.g., neglecting the thermal conductivity and shear viscosity), we can obtain from the hydrodynamics discussed in this section,

$$T\partial_{\mu}S^{\mu} = \delta\mu\Delta\Gamma = \lambda\delta\mu^2. \quad (89)$$

A similar expression is valid for npe matter [see Eq. (79) of Ref. [17]]. The chemical potential disbalance $\delta\mu$ is given by Eq. (59). Substituting into Eq. (59) variations δn_b, δn_H, and $\delta n_{\Sigma n}$ calculated from the continuity equations (56)–(58), we find

$$\delta\mu = \frac{1}{\lambda(\partial\delta\mu/\partial n_H - \partial\delta\mu/\partial n_{\Sigma n}) - i\omega}\left[\frac{\partial\delta\mu}{\partial n_b}\mathrm{div}(\boldsymbol{J}_b) + \frac{\partial\delta\mu}{\partial n_H}\mathrm{div}(\boldsymbol{J}_H) + \frac{\partial\delta\mu}{\partial n_{\Sigma n}}\mathrm{div}(\boldsymbol{J}_{\Sigma n})\right]. \quad (90)$$

It follows from Eq. (90), that for any given u^{μ} it is always possible to choose four vectors $w^{\mu}_{(i)}$ in such a way that $\delta\mu = 0$ at some point and in some particular moment (even if some baryon species are nonsuperfluid). In other words, this means that we can vanish the entropy generation rate (89) at this point.

On the other hand, the entropy generation equation in terms of the effective bulk viscosities takes the form [see Eqs. (38) and (40)–(44)]

$$\begin{aligned} T\partial_{\mu}S^{\mu} = {} & \{\xi_{1q}\partial_{\mu}[Y_{qk}w^{\mu}_{(k)}] + \xi_2\partial_{\mu}u^{\mu}\}\partial_{\mu}u^{\mu} \\ & + \{\xi_{3q}\partial_{\mu}[Y_{qk}w^{\mu}_{(k)}] + \xi_{4n}\partial_{\mu}u^{\mu}\}\partial_{\mu}[Y_{nk}w^{\mu}_{(k)}] \\ & + \{\xi_{5q}\partial_{\mu}[Y_{qk}w^{\mu}_{(k)}] + \xi_{4\Lambda}\partial_{\mu}u^{\mu}\}\partial_{\mu}[Y_{\Lambda k}w^{\mu}_{(k)}] \\ & + \{\xi_{6q}\partial_{\mu}[Y_{qk}w^{\mu}_{(k)}] + \xi_{4\Sigma p}\partial_{\mu}u^{\mu}\}\partial_{\mu}[Y_{\Sigma pk}w^{\mu}_{(k)}]. \end{aligned} \quad (91)$$

The right-hand side of this equation satisfies two conditions. First, it is a positive-definite quadratic form (entropy cannot decrease). Second, according to Eqs. (89) and (90), one can vanish it by an appropriate choice of u^{μ} and $w^{\mu}_{(i)}$ (at some particular moment and at some point). There is a mathematical theorem that these two conditions are consistent with each other if and only if the determinant of the matrix composed of coefficients of the quadratic form vanishes. This result is independent of an actual number of superfluid baryon species. That is, the determinant will be zero in the case when all four baryon species are superfluid as well as in the case when some of them are normal (for example, nucleons are normal, hyperons are superfluid, or neutrons are superfluid, while other particles are normal). As a consequence, we arrive at the six additional constraints (84) and (85) on the bulk viscosity coefficients.

In this section, we have assumed that baryons of all species are superfluid. However, the hydrodynamics formulated here can be easily extended to the situation when some baryon species are normal. In this case, one should vanish matrix elements Y_{ik} related to these baryon species. For example, if neutrons are nonsuperfluid, then $Y_{nk} = Y_{kn} \equiv 0$.

IV. SUMMARY

In this paper, we have analyzed the bulk viscosity due to nonequilibrium particle transformations in superfluid nucleon-hyperon matter of neutron stars. Our approach is similar to that used in Ref. [17] for the case of superfluid npe matter.

Our main results are as follows:

(i) We have demonstrated that the expression for the bulk viscosity of normal (nonsuperfluid) nucleon-hyperon matter, widely used in the literature, is inaccurate. We have presented the correct derivation of the bulk viscosity and compared it with the results of Ref. [14]. Numerically, both formulas give almost similar results (within a few tens of percent).

(ii) We have extended the hydrodynamics of superfluid mixtures reported in Refs. [17,31] to allow for a possible presence of superfluid hyperons. We have determined the general form of dissipative terms entering the equations of this hydrodynamics and showed that generally (when all baryon species are superfluid), it contains *16* bulk viscosity coefficients.

(iii) We have calculated and analyzed the 16 bulk viscosity coefficients assuming they are generated by nonequilibrium reactions (1)–(4) of particle mutual transformations. We have shown that only *three* of them are independent. All other 13 bulk viscosities can be expressed through these three using Eqs. (50) and (83)–(85). In addition, we have demonstrated that Eq. (67) for the bulk viscosity coefficient ξ_2 formally coincides with Eq. (22) for that in normal matter [however, the reaction rates λ_l ($l = 1, \ldots, 4$) are affected by superfluidity].

Our results can be important for the studies of dynamical instabilities in pulsating superfluid neutron stars, especially for the studies of the r mode instability. They can also be important for modeling of the thermal evolution of pulsating neutron stars and for analyzing rotochemical and gravitochemical heating of millisecond pulsars with superfluid nucleon-hyperon cores (for nonsuperfluid npe matter of neutron stars, this problem was considered in Refs. [28,35–37]).

Let us notice that to start such an analysis one needs to know the matrix Y_{ik}, which is the most important ingredient in the hydrodynamics of superfluid nucleon-hyperon mixture. To our best knowledge, it has not been considered in the literature so far. We are planning to fill this gap and present its extensive calculations in a subsequent publication.

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